Extensional concepts in intensional type theory, revisited

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In his Ph.D. dissertation, Hofmann [Hof95; Hof97] constructs an interpretation of extensional type theory in intensional type theory, subsequently proving a *conservativity* result of the former over the latter extended by the principles of functional extensionality and of uniqueness of identity proofs (UIP), cf. [Hof95, §3.2]. Interestingly, Hofmann's proof is "stronger" than the statement of his theorem, as the requisite language in which to speak of conservativity and equivalence of dependent type theories did not exist at the time.

A major insight came as a result of the development of homotopy type theory. In particular, Voevodsky's definition, ca. 2009, of when a morphism in a model of dependent type theory is a *weak equivalence* allows one to consider different homotopy-theoretic structures both within such models and the categories thereof.

In [KL18], it was observed that the category of models of a dependent type theory carries the structure of a left semi-model category. This structure was subsequently used by Isaev [Isa18] to define a *Morita equivalence* of dependent type theories. In essence, two theories are Morita equivalent if their categories of models are equivalent in a suitable "up-to-homotopy" sense. More precisely, a Morita equivalence between theories is a translation between them that induces a Quillen equivalence (the correct notion of equivalence for left semi-model structures) between their left semi-model categories of models. Perhaps unsurprisingly, Isaev cites Hofmann's theorem as one of the motivating examples behind his definition, without actually proving it to be one.

In our recent publication [KL25], we give a direct proof of Morita equivalence between the extensional type theory and the intensional type theory extended by the principles of functional extensionality and of uniqueness of identity proofs. While Hofmann proves that the initial models of these theories are suitably equivalent, we generalize this result to all possible extensions of the base theories by types and terms, including propositional equalities. In homotopy-theoretic terms, these are exactly the *cofibrant* extentions.

Therefore, thanks to proving Morita equivalence, one does not need to prove an analogue of Hofmann's result for any new extension but instead appeal to our result addressing all extensions once and for all. As new variants and extensions of intensional type theory are constantly proposed, this reduces the burden of proving their expected properties by making what should be formal formal.

Our proof follows Hofmann's quite closely but requires a major innovation to account for all possible (cofibrant) extensions. In [Hof97], Hofmann describes a class of context morphisms termed *propositional isomorphisms* by inspecting the outermost type former in (the last type of) each context and collapses these maps to identities, thus obtaining a functor from the syntactic model of intensional type theory to that of extensional type theory.

Our general approach is nearly identical: first, we describe the *extensional kernel* of a cofibrant model of intensional type theory, which we then collapse in a construction reminiscent of Hofmann's to obtain a (cofibrant) model of extensional type theory. This explicit construction then allows one to directly verify that it produces a model of extensional type theory and yields Morita equivalence between the theories. Because of working with syntactic categories, Hofmann is able to inspect the outermost constructor in each context. This property need not hold in every model, but it does hold in the cofibrant ones, which is therefore sufficient to follow the remainder of Hofmann's strategy to establish a Morita equivalence.

In this talk, after reviewing the necessary background, we give an overview of the proof, including a detailed comparison with Hofmann's approach.

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