Introducing Displayed Universal Algebra in UniMath

Gianluca Amato*, Matteo Calosci°, Marco Maggesi°, and Cosimo Perini Brogi⋆

*Università di Chieti-Pescara
°Università di Firenze
⋆Scuola IMT Alti Studi Lucca

General framework

Both universal algebra and category theory provide unifying frameworks for studying algebraic structures in abstract terms [8, 7, 1].

In universal algebra, a structure – be it a group, a ring, or a lattice – is described as a theory, i.e. in terms of abstract operations over a carrier and a set of equations these operations are required to satisfy. This way, once a theory is specified, specific instances of algebraic structures are defined as models of that theory, and an algebra homomorphism is a structure-preserving map between models [12].

In category theory, Lawvere theories provide a presentation-invariant version of algebraic theories, where algebraic operations are given by morphisms in the Lawvere theory. Specific instances of algebraic structures are defined as product-preserving functors from a Lawvere theory to categories with finite products. This reveals universal algebra as the theory of all structures that can be defined in categories with finite products [11].

Now, in universal algebra, it is often possible to modularly construct an algebraic structure. Similarly, the work started with [2] showed that many categorical constructions can be developed modularly both at the objects and morphisms levels (as well as at higher categorical levels [3]), by adding progressively layers of further structure. In this work we show how to rephrase the methods of displayed categories in order to deal with classical concepts of universal algebra, making the analogy we have just sketched more precise. We thus widen the investigations on the notion of *displayed algebras* [9, 10, 13], with a focus on developing universal algebra with the displayed-category style.

Concretely, given a base algebra, we *display* the additional algebraic structure – elements and operations – in a structured layer above such base. This approach not only streamlines proofs of properties and constructions but also facilitates the reuse of generic lemmas across different algebraic contexts. This way we can reap the organizational benefits of displayed formalisms both for the mathematical developments of universal algebra and their computer implementations.

Source code. Building on our previous work [4, 5], our current results are fully mechanized in the UniMath library[1], so that all our definitions and theorems are consistently formalized within univalent mathematics. UniMath offers a convenient and natural foundation for implementing various branches of universal algebra and demonstrates how the displayed methodology can be used to uniformly build robust and modular constructions both at the categorical level and in the more traditional algebraic setting.

Our code is freely available from our GitHub repository[2] and it is in the process of being integrated in the UniMath library. When discussing concepts which are available in our source code, we will provide specific links below.

[1] `https://github.com/UniMath/UniMath`
[2] `https://github.com/UniMathUA/UniMath/tree/hott-uf-2025`

Main contents and current development

The natural starting point of our work is the formal definition of displayed algebras.

Definition. *Given an algebra $\mathcal{B}$ over a multi-sorted signature σ, a displayed algebra over $\mathcal{B}$ consists of:*

- *a family of "fiber" types indexed over terms of $\mathcal{B}$;*
- *a family of functions indexed over any operation name f of σ and any vector v of terms of $\mathcal{B}$ each having the appropriate sort specified by the arity of f. Each function has the product of the fiber of the components of v as domain and the fiber of $f^{\mathcal{B}}(v)$ as codomain.*

In the categorical setting, a displayed category encodes the same information as a functor into the base category. In our algebraic setting, we show the analogous result :

Theorem. *Given an algebra $\mathcal{B}$ over a multi-sorted signature σ, the type of algebra morphisms targetting $\mathcal{B}$ is equivalent to the type of displayed algebras over $\mathcal{B}$.*

In a straightforward and natural way, every displayed category gives rise to a total category. In the same spirit, every displayed algebra gives rise to a total algebra . This construction is, in fact, one of the primary motivations for working with displayed structures in the first place. One of our objectives is to provide a clear, modular framework for assembling algebras from simpler components.

Motivating Examples. We emphasize how various familiar algebraic constructions – in particular, those admitting a staged or layered description – can be systematically recovered by taking the total algebra of a displayed algebra. Here are some examples:

Cartesian Products: A displayed algebra can encode how algebraic operations behave componentwise on a product of carriers. Collecting these componentwise structures into a single object yields the familiar cartesian product algebra as its total algebra.

Pullbacks: By generalizing the previous example, one can display algebraic structure along pullback squares to obtain a pullback of algebras. The resulting total algebra thus inherits its operations from the displayed structure, reflecting the universal property of the pullback on the level of algebras.

Semidirect Products: The semidirect product of groups can also be viewed as the total algebra of a suitable displayed algebra. Here, one "displays" how a normal subgroup and a quotient group interact, and then reassembles this information into the total structure defining the semidirect product.[3]

Subalgebras: Consider a displayed algebra that restricts the underlying carriers of a larger algebra to subsets closed under its operations. When one collects this restricted (or "sub") structure into a single object, the total algebra precisely captures the notion of a subalgebra .

Related and future work

Our novel formalism for displayed algebras is inspired from the work in displayed categories started with [2]; in future, we plan to explore further connections between their displaying methods for categories and our constructions at the algebraic level. At the same time, we plan to extend our original UniMath library for universal algebra [5] by making extensive use of the techniques of displayed algebras we communicate here. Finally, it would be relevant to bridge our formalization of universal algebra in UniMath with the existing library on Lawvere theories[4] in the same univalent system and investigate potential transfers of results between the categorical and universal languages for algebraic structures using a unified displayed formalism.

[3] While semidirect products of groups have well-known generalizations to other contexts in universal algebra [6], those generalizations are not treated here.

[4] https://github.com/UniMath/UniMath/tree/master/UniMath/AlgebraicTheories

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