The Cantor–Schröder–Bernstein Theorem in ∞-topoi

Fredrik Bakke
Norwegian University of Science and Technology

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Abstract

We present work-in-progress generalizations of the Cantor–Schröder–Bernstein theorem that applies in non-boolean ∞-topoi.

The Cantor–Schröder–Bernstein theorem is a fundamental result in classical set theory: given two sets A and B such that A injects into B and B injects into A, then there is a bijection $A \simeq B$. It has long been known that the theorem can be proven assuming the law of excluded middle. More recently, Brown and Pradic [PB22] established the converse as well: the law of excluded middle follows from the Cantor–Schröder–Bernstein Theorem. Their proof uses the "logical compactness" of the extended natural numbers $\mathbb{N}_\infty$, a result of Escardó [Esc13]. Moreover, Escardó has given a generalization of the Cantor–Schröder–Bernstein Theorem in Martin–Löf type theory that applies to arbitrary ∞-groupoids [Esc21], or indeed in arbitrary boolean ∞-topoi following work by Shulman [Shu19]:

Theorem (Escardó [Esc21])
Assuming function extensionality and the law of excluded middle, any two types A and B such that A embeds into B and B embeds into A are equivalent.

In this work-in-progress, we consider generalizations of this theorem that apply to more general classes of ∞-topoi, by adding further conditions on the embeddings that otherwise follow from the law of excluded middle. The results are formalized in the agda-unimath library [RSPB+]. At the moment of writing, we have established and formalized the following generalization of the theorem.

Theorem
Assuming function extensionality and Bishop's weak limited principle of omniscience (universal quantifications over decidable predicates on $\mathbb{N}$ are decidable), then any two types A and B such that A embeds into B and B embeds into A via embeddings with decidable fibers, then A and B are equivalent.

However, we expect some iteration of the following stronger result to be true:

Conjecture
Assuming function extensionality, if two types A and B mutually embed into each other via embeddings with decidable fibers, then A and B are equivalent.

While the details remain to be worked out, we take inspiration from the works of Escardó [Esc13], Blechsmidt–Oldenziel [BO], and Rijke–Shulman–Spitters [RSS20], among others, and investigate appropriate compactness and continuity conditions for the setup.

Comment. Forster, Jahn, and Smolka have considered another constructive generalization of the Cantor–Schröder–Bernstein Theorem [FJS23] that applies to types A and B that are retracts of $\mathbb{N}$. We emphasize that the present work considers a markedly different generalization that applies to arbitrary, or at least much larger classes of types, and instead adds further conditions on the maps.

References

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