Computable and non-computable 2-groups in HoTT

Andrew W Swan

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In the classical approach to computability theory we think of the collection of computable functions as a subset of the larger class of all functions $\mathbb{N} \to \mathbb{N}$. The larger class also contains some functions that are not computable, with the simplest being the characteristic function of the halting set. In synthetic computability theory [Ric83, Bau06] we can view the same results from a completely different perspective. By default we assume that everything is computable, using axioms such as Church's thesis, which states that every function $\mathbb{N} \to \mathbb{N}$ is computable. Even though functions $\mathbb{N} \to \mathbb{N}$ are all computable, we can still have access to non computable functions by choosing our definition appropriately. One way [Swa24] is to use higher modalities [RSS20], such as ∇, the modality of double negation sheaves: we still have the halting set, but instead of viewing it as a function $\mathbb{N} \to 2$ that does not belong to subset of computable functions, we view it as a function $\mathbb{N} \to \nabla 2$ that does not factor through the inclusion $2 \hookrightarrow \nabla 2$.

The synthetic approach has a key advantage when working with *computable structures* [Rab60]. An algebraic structure is *computable* when its carrier set is a computably decidable subset of $\mathbb{N}$ and all algebraic operations are computable. Working synthetically we can make a simplification to this definition: an algebraic structure is computable when its carrier set is a decidable subset of $\mathbb{N}$. We can drop the requirement that algebraic operations are computable, since this is automatically the case by default, using Church's thesis. To talk about structures with non computable algebraic operations, we need to switch to a different carrier set, for example by applying ∇.

We develop a synthetic approach to computable 2-groups based on the elegant characterisation of higher groups in HoTT as simply pointed connected types [BvDR18]. In particular, we can define 2-groups as pointed, connected, 2-truncated types. We say a 2-group $(BG, \mathtt{base})$ is *computable* when $\|\Omega(BG, \mathtt{base})\|_0$ and $\Omega^2(BG, \mathtt{base})$ are both equivalent to decidable subsets of $\mathbb{N}$. A result due to Sính [Sín75, BL04] shows that 2-groups can be understood algebraically via the following structure:

1. A group G.

2. An abelian group H.

3. An action of G on H.

4. An element of the cohomology group $H^3(G, H)$[1] (the *Sính invariant* of the 2-group)

Milner [Mil24] has developed a purely HoTT version of this result, showing how to derive all of this structure from a pointed, connected, 2-truncated type.[2] It follows by Church's thesis that given the simple definition of computable 2-group above, all of the resulting algebraic structure is computable, making this definition a natural generalisation of the classical definition of computable group as in [Rab60].

After introducing the above ideas, I will give some purely synthetic constructions of non computable 2-groups, including a finitely generated 2-group which is non computable but has computable underlying 1-group. The proof makes use of a non-computable action of the 2-group on groupoids, which combines the Buchholtz-Van Doorn-Rijke approach to group theory with ideas from synthetic computability theory, in particular the use of ∇ to allow the action to be non-computable.

References

[Bau06] Andrej Bauer. First steps in synthetic computability theory. *Electronic Notes in Theoretical Computer Science*, 155:5–31, 2006. Proceedings of the 21st Annual Conference on Mathematical Foundations of Programming Semantics (MFPS XXI).

[BL04] John C. Baez and Aaron D. Lauda. Higher-dimensional algebra. V: 2-groups. *Theory and Applications of Categories*, 12:423–491, 2004.

[BvDR18] Ulrik Buchholtz, Floris van Doorn, and Egbert Rijke. Higher groups in homotopy type theory. In *Proceedings of the 33rd Annual ACM/IEEE Symposium on Logic in Computer Science*, LICS '18, page 205–214, New York, NY, USA, 2018. Association for Computing Machinery.

[Mil24] Owen Milner. 2-Groups in HoTT, 2024. Unpublished note.

[Rab60] Michael O. Rabin. Computable algebra, general theory and theory of computable fields. *Transactions of the American Mathematical Society*, 95(2):341–360, 1960.

[Ric83] Fred Richman. Church's thesis without tears. *Journal of Symbolic Logic*, 48(3):797–803, 1983.

[RSS20] Egbert Rijke, Michael Shulman, and Bas Spitters. Modalities in homotopy type theory. *Logical Methods in Computer Science*, Volume 16, Issue 1, January 2020.

[Sín75] Hoàng Xuân Sính. *Gr-categories.* PhD thesis, Institut Pédagogique no. 2 de Hanoi, 1975.

[1] with H viewed as a module over G

[2] He showed moreover that replacing the definition of cohomology with an "untruncated" version in fact makes this an equivalence of types.

[Swa24] Andrew W Swan. Oracle modalities, 2024. Preprint available arXiv:2406.05818.