Towards a type theory for (∞, ω)-categories

Louise Leclerc

March 13, 2025

The subject is to present the ideas underlying *Cellular Type Theory (CellTT)*, a type theory I am working on. The aim is to have a theory akin to *Simplicial Type Theory* (STT) of Riehl and Shulman [3], but where types should be thought as presheaves over the category of cells Θ, instead of Δ. So CellTT should be to Θ-spaces [2] what STT is to Segal spaces. Note that this is very much work in progress as it is the subject of my ongoing thesis, supervised by Mimram. Most of the mathematical theory underlying the definitions given in the next section are compiled in Loubaton's Thesis [1].

1 Main ideas

1.1 Differences with the $(\infty, 1)$-case. In Simplicial Type Theory, hom types are discrete, in the sense that they are ∞-groupoids. This occurs because Rezk-types are "only" $(\infty, 1)$-categories. While in the (∞, ω)-setting, hom types of (∞, ω)-categories should remain (in general, non-discrete) (∞, ω)-categories. This crucial difference allows the authors of STT to define hom types as usual mapping spaces

$$\mathsf{hom}_A(x, y) = \sum_{f : I \to A} f(0) = x \times f(1) = y$$

where I - a directed interval - is interpreted as the Yoneda embedding of the 1-simplex (seen as an object of Δ, in the $(\infty, 1)$-case, and as an object of Θ, in the (∞, ω)-case). Howether, when working with (∞, ω)-categories, this mapping type behaves differently to the hom type. A 2-cell in A should be a map $I \to (I \to A)$ with boundary conditions, which by Curryfication, should be the same as a map $I^2 \to A$ with boundary conditions. Whereas I^2 is only a 1-categorical object. Hence, we must find a way to define the hom type differently.

1.2 Introducing $\flat$. One way to do so is to work with an idempotent modality which is comonadic, for instance building upon *Crisp Type Theory*, as introduced by Shulman in [4]. Here, our flat modality comes from an adjunction :

$$\mathcal{S} \underset{\mathsf{ev}_* = (F \mapsto F(*))}{\overset{\delta}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} [\Theta^{\mathrm{op}}, \mathcal{S}]$$

where $\mathcal{S}$ is the $(\infty, 1)$-categeory of spaces, and $\delta(X)$ is the constant functor equal to X. We then have an idempotent comonad $\flat = \delta \circ \mathsf{ev}_* : [\Theta^{\mathrm{op}}, \mathcal{S}] \to [\Theta^{\mathrm{op}}, \mathcal{S}]$. This setting falls in the broader one of local toposes which should be model of the *spatial* fragment of cohesive HoTT.

This modality allows us to speak about the types of "points" X_* of a type X by seing it as a discrete (i.e. constant) presheaf (i.e. type). When working with a type which is an (∞, ω)-category, it should computes its core.

1.3 Pasting schemes. We define inductively the type of *pasting schemes* PS, which is a set, and corresponds to the objects of the category Θ. It has a unique constructor $\mathsf{cons} : \mathsf{PS\,List} \to \mathsf{PS}$. We denote succintly $[P_1, \cdots, P_n]$ the pasting scheme $\mathsf{cons}[P_1, \cdots, P_n]$, and $\$: \mathsf{PS} \to \mathsf{PS}$ the operation (of suspension) $P \mapsto [P]$.

Then $[n]$ denotes the list $[[], [], \cdots, []]$ of length n, and $\mathcal{O}_n = \$^n[]$ is the list $[[\cdots [] \cdots]]$ containing $n+1$ pairs of brackets.

One may also define types $\mathsf{Hom}(P, Q)$ for the morphisms of Θ between $P, Q : \mathsf{PS}$, which are sets too.

1.4 Yoneda embedding and suspension. Then we postulate the existence of a Yoneda embedding $\|-\| : \mathsf{PS} \to \mathcal{U}$ (which may be thought as the (∞, ω)-category associated to a pasting scheme), together with a suspension opration $\$' : \mathcal{U} \to \mathcal{U}_{\bullet,\bullet}$ extending to all types the suspension of pasting schemes $\$: \mathsf{PS} \to \mathsf{PS}$. where $\mathcal{U}_{\bullet,\bullet}$ denotes the type of bipointed types. We let $\mathbb{1} :\equiv \|\Delta^0\|$, $I :\equiv \|\Delta^1\| \equiv \|\mathcal{O}_1\|$ and $\mathsf{D}_n :\equiv \|\mathcal{O}_n\|$. A type A is said to be *discrete* iff for all $P : \mathsf{PS}$, the canonical map $A \to (\|P\| \to A)$ is an equivalence. And we axiomatize that this is the same as having the counit $\flat A \to A$ being an equivalence. This axiom will be called *cellular cohesion*.
Now, for each type, we may consider its P-cells for any crisp $P : PS$: $X_P :\equiv \flat(\|(\|P) \to X)$. Moreover, each crisp map $f : X \to Y$ will induce maps $X_P \to Y_P$ for each P. By postulating that objectwise equivalences are equivalences, we may show that $\mathbb{1}$ is contractible, or that equality of maps are given by objectwise equalities.

1.5 Hom types. For a type A and $x, y : A$, we postulate a type $\mathsf{hom}_A(x, y)$ together with the *probing principle*

$$(\|P\| \to \mathsf{hom}_A(x, y))^\flat \cong (\|\$P\| \to_{\bullet,\bullet} (A, x, y))^\flat$$

naturally in P and functorially in A. This way, the hom type is seen as coming from an adjunction with the suspension

$$[\Theta^{\mathrm{op}}, \mathcal{S}] \underset{\mathsf{hom}}{\overset{\$}{\rightleftarrows}} [\Theta^{\mathrm{op}}, \mathcal{S}] \qquad (\$ \dashv \mathsf{hom})$$

1.6 (non-fibrant) Realization of a pasting scheme. One may define a realization of any pasting scheme $P \equiv [P_1, \cdots, P_m]$, denoted $|P|$, as the following colimit of types.

$$|[P_1, \cdots, P_m]| :\equiv$$

$$\mathsf{colim}\left(\$'|P_1| \leftarrow \mathbb{1} \rightarrow \$'|P_2| \leftarrow \mathbb{1} \rightarrow \cdots \leftarrow \mathbb{1} \rightarrow \$'|P_m| \right)$$

1.7 Segal types. There is a canonical map $|P| \to \|P\|$, and one say that a type A is a Segal-type, whenever for each $P : \mathsf{PS}$ the following maps (induced by precomposition) are aquivalences

$$(\|P\| \to A)^\flat \to (|P| \to A)^\flat \quad .$$

1.8 (∞, ω)-categories. We should also have a notion of completeness, in the same sense as in simplicial type theory. And types which should be thought as (∞, ω)-categories should correspond to the one that are Segal and complete.

2 What is to be done

Once the definition of *Segal* and complete types have been given, it makes sense to ask if a type is an infinity category or not. For instance, one has very formally that a product of Segal types is Segal, or that the unit type is Segal. What is harder, and is one of the main goals of my ongoing work, is to prove a Yoneda lemma in this setting.
This requires having a well-suited definition of fibration (because I cannot hope for a directed univalent universe at the moment), and to prove that $\mathsf{hom}_A(-, a)$ is an (op)fibration when A is an (∞, ω)-category.

A more accessible result which I'm working on at the moment is the property that whenever a type is Segal, its hom type should be a Segal type too. This seems possible, although not straitforward to formalize (in $\flat$-AGDA). And I hope to have established this result by the time of my presentation at HoTT-UF 2025.

References

[1] Félix Loubaton. Theory and models of (∞, ω)-categories, 2023.

[2] Charles Rezk. A cartesian presentation of weak n-categories. *Geometry and Topology*, 14(1):521–571, January 2010.

[3] Emily Riehl and Michael Shulman. A type theory for synthetic ∞-categories, 2023.

[4] Michael Shulman. Brouwer's fixed-point theorem in real-cohesive homotopy type theory, 2017.