Projective Presentations of Lex Modalities

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In constructive mathematics, and the internal language of topoi, constructing modalities is a common tool. For instance, in [2, Section 8] an internal model of synthetic algebraic geometry is constructed. Their method is to begin with a simple axiomatic setting, describing a presheaf category: the classifying topos of the theory of rings. Then, they define a modality corresponding to an ordinary external Grothendieck topology, and it is shown the subuniverse described by this modality satisfies the axioms of synthetic algebraic geometry. In [3], a similar method is used to carve out the category of simplicial spaces out of a larger presheaf topos.

In the present work [10], we describe a systematic framework for describing similar modalities across various axiomatic systems. In this framework we show internal sheaf conditions, a useful tool for proving when an n-truncated type lives in a subuniverse. Given a further condition on the presentation, that of projectivity, we give a descent condition for sheaf cohomology. We apply this descent condition to synthetic algebraic geometry to replicate a well known result from algebraic geometry: calculating the cohomology of quasi-coherent sheaves in several topoi.

Modalities The general theory of (monadic, idempotent) modalities has been investigated by Spitters, Shulman, and Rijke [8]. There, a modality is described by an operator on types $\bigcirc : \mathcal{U} \to \mathcal{U}$, with a family $\eta : (A : \mathcal{U}) \to A \to \bigcirc A$, satisfying a modal induction principal. Not all modalities correspond a subtopos, only the left exact ones, which are called **lex modalities**. Further, they distinguish a class of modalities externally corresponding to localisations via a Grothendieck topology. Given a family $P : I \to \mathrm{Prop}_{\mathcal{U}}$, a type X is called a P**-sheaf** if for all $i : I$, the diagonal $X \to (P(i) \to X)$ is an equivalence. From this family, we can form a modality $\bigcirc_P$, naturally satisfying that $\bigcirc_P X$ is P-null for all $X : \mathcal{U}$. A modality of this form is called **topological**.

Presentations Following Moeneclaey [5], we encode of the notion of Grothendieck topology internally: A **presentation** is a collection T of types so that $1 \in T$, which is closed under Σ. Given such a family, the modality it presents is nullification at the family $\{\|X\| \mid X \in T\}$, where $\|X\|$ is the propositional truncation. We call a sheaf for this family a T-sheaf, and denote sheafification by $\bigcirc_T$. A map $f : A \to B$ is a T**-cover** if for all $b : B$, the fiber $\mathrm{fib}_f(b)$ is in T.

Sheaf conditions Given a presentation, we prove a novel internal version of (higher) sheaf conditions. Given a map $f : A \to B$ and $n \geq 0$, we can form its n-fold iterated join power $A^{*_B n}$ [6]. Our result is then:

Theorem 1 (Sheaf condition) Let T be a presentation and X be an n-type for $n \geq -2$. Then X is a T-sheaf iff for all covers $f : A \to B$ the natural map $(B \to X) \to (A^{*_B (n+2)} \to X)$ is an equivalence.

Given a concrete natural number, say $n = 0$, this reduces to a more recognisable sheaf condition:

Corollary 2 A 0-type X is a T-sheaf iff for all covers $f : A \to B$ the natural map

$$X^B \to \lim \left(X^A \rightrightarrows X^{A \times_B A} \right)$$

is an equivalence.

Projective presentations Externally, a presentation is understood as forming a Grothendieck topology on a presheaf category, by choosing a coverage of representable functors. As representable functors are tiny, they are internally projective, that is, they satisfy the internal axiom of choice. We call a presentation T **projective** if all $X \in T$ are projective. Given such a presentation, we internally and constructively describe a method to calculate first cohomology of quasi-coherent sheaves in several toposes, generalising the proof of [9, Tag 03P2].

Given a presentation T, a T-sheaf X and a group T-sheaf G, for each $n \geq 0$ we define the modal sheaf cohomology to be

$$H^n_T(X, G) := \bigcirc_T \| X \to \bigcirc_T K(G, n) \|_0$$

where $K(G, n)$ is the n^{th} Eilenberg-MacLane space [4] and $\|\_\|_0$ is set truncation.

We define an abelian group T-sheaf A to satisfy **descent** for T if for all $X \in T$ the sequence

$$A^X \xrightarrow{d^0} A^{X \times X} \xrightarrow{d^1} A^{X \times X \times X}$$

is exact, where $d^0(f)(x, x') := f(x) - f(x')$ and $d^1(f)(x, x', x'') := f(x, x') - f(x, x'') + f(x', x'')$.

Theorem 3 Let T be a projective presentation, and A be an abelian group T-sheaf satisfying descent. Then for all projective X we have $H^1_T(X, A) = \bigcirc_T 0$.

As a corollary we are able to show that for the fppf, étale and Zariski topoi, first cohomology of affine schemes is zero on quasi-coherent modules, as these satisfy descent for the resepctive topologies. Higher cohomologies are also zero, which can be bootstrapped from first cohomology using the results of Blechschmidt, Cherubini, and Wärn [1].

Application For a generic setting featuring interesting presentations, we introduce a simple axiomatic system, parametrised by a choice of algebraic theory $\mathbb{T}$, with a model in the classifying topos in $\mathbb{T}$. These axioms are precisely analogous to those of the internal sheaf model in section 8 of [2]. We then build projective presentations in this system.

By specialising $\mathbb{T}$ to the theory of rings, we obtain the same axiomatisation used in [2], and can derive the same results they do, but with our methods. We also establish the novel internal proof of the cohomology of quasi-coherent sheaves.

If $\mathbb{T}$ is the theory of bounded distributive lattices, our axioms are similar to triangulated type theory [3]. In particular there is a bounded distributive lattice $\mathbb{I}$, taking the role of the interval. In this setting we define a presentation for the simplicial modality. We apply the sheaf condition to prove directly that the interval $\mathbb{I}$ is a simplicial type in plain type theory, in contrast to the proof given by [3], which requires a modal extension of HoTT.

Future Work We hope this work will contribute to the methods used throughout synthetic mathematics in HoTT. Additionally, we hope to apply the axioms for the internal logic of the classifying topos of an algebraic theory to other domains, where similar approaches have been taken, such as Simpson's random topos [7].

References

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