Data Types with Symmetries via Action Containers

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Containers are a representation of strictly positive data types, introduced by Abbott et al. [1]. They consist of a type of *shapes* and a type of *positions* associated to each shape, and are interpreted as polynomial endofunctors, modelling the polymorphic data types they represent. Natural transformations of these functors model polymorphic functions of data types, and interpretation of containers is functorial: Containers form a category in which morphisms faithfully represent polymorphic functions.

Traditionally, the theory of containers is studied in Set-like categories. When interpreted in such categories, the data of containers may however be too restrictive to encode certain data types of interest. This is especially the case if one wants to account for symmetries, i.e. identify configurations of positions when one can turn into the other via the action of certain permutations. For example, one can represent ordered lists, but it is not possible to represent cyclic lists or finite multisets as a container.

Some efforts have been made to enhance the expressivity of containers to represent data types with symmetries. Abbott et al. [2] introduced quotient containers, which are containers in which the assignment of values to positions is invariant under a group of permutations on positions. Interpretation of quotient containers embeds them as a subcategory of set-endofunctors, targeting certain quotients of polynomial endofunctors, typically called analytic functors [9].

Symmetric containers, introduced by Gylterud [7], consist of a groupoid of shapes S and a set-valued functor $P : S \to \mathsf{Set}$ of positions. Symmetries are encoded directly in the isomorphisms of the shape groupoid, and are mapped, functorially, to permutations of sets of positions. From the perspective of homotopy type theory, symmetric containers correspond to families of positions $P : S \to \mathsf{hSet}$ over shapes $S : \mathsf{hGpd}$. Symmetric containers form a locally univalent 2-category $\mathsf{SymmCont}$, and can be interpreted as polynomial endofunctors on the 2-category of groupoids.

Quotient and symmetric containers are two different ways to extend the expressivity of ordinary containers to include symmetries between positions. To understand how these two approaches are related, we introduce an intermediate notion: *action containers*.

Definition. An action container $(S \triangleright P \triangleleft^{\sigma} G)$ consists of a set of shapes S and, for each shape s, a set of positions P_s, a group G_s, and an action σ_s of G_s on P_s.

Example. The container $\mathsf{Cyc} := (\mathbb{N} \triangleright \mathsf{Fin} \triangleleft^{\sigma} \mathbb{Z})$ has $\mathbb{Z}$ acting on $\mathsf{Fin}(n)$ as follows: for each n, let $\sigma_n : \mathbb{Z} \to \mathfrak{S}(n)$, $\sigma_n(k) := \lambda \ell.\, (\ell + k) \mod n$. This container represents *cyclic lists*: lists of n elements are identified up to a cyclic shift by $k : \mathbb{Z}$ positions.

On one side, such containers generalize quotient containers, as the allowed permutations are determined by the action of an arbitrary group, and are not restricted to subgroups of symmetric groups of positions. On the other, they are a special case of symmetric containers: a G_s-action is a functor from G_s (seen as a 1-object groupoid) to Set, and summation of these functors over all shapes s yields a symmetric container.

Morphisms of action containers, unlike those of quotient containers, have to explicitly preserve the structure of the groups acting on positions. We show that the category of action containers is freely generated as a category of families, from which we easily derive closure properties:

Theorem. *The category of action containers is the free coproduct completion of a category of group actions and equivariant maps.*

Proposition. *Action containers are closed under arbitrary products, coproducts, and exponentials with constant domain.*

We show that action containers are a well-behaved subclass of symmetric containers by defining a locally fully faithful 2-functor into $\mathsf{SymmCont}$. We construct the 2-category of action containers in a modular fashion: we start from a 2-category Group of groups, group homomorphisms and *conjugators* [8]. On this we define a 2-category of group actions, using the techniques of displayed bicategories [4]. We then repeat a 2-categorical version of the Fam-construction, presenting the 2-category of action containers as that of families of group actions, $\mathsf{ActCont} := \mathrm{Fam}(\mathsf{Action})$.

We observe that symmetric containers are exactly set bundles over homotopy groupoids (in the sense of [5, §3.3]). Identifying the 2-categories of set bundles and symmetric containers, we construct a 2-functor $\mathsf{ActCont} = \mathrm{Fam}(\mathsf{Action}) \to \mathsf{SetBundle} = \mathsf{SymmCont}$ in multiple steps. Just like the 1-category of groups is equivalent to the category of pointed connected groupoids, Group is equivalent to the 2-category of connected groupoids. Thus, delooping of a group extends to a 2-functor $\mathbf{B} : \mathsf{Group} \xrightarrow{\sim} \mathsf{hGpd}_{\mathsf{conn}} \hookrightarrow \mathsf{hGpd}$, which is locally a weak equivalence. Using displayed machinery, we lift this to a local weak equivalence $\bar{\mathbf{B}} : \mathsf{Action} \to \mathsf{SetBundle}$ of 2-categories.

The Fam-construction yields a 2-functor $\mathrm{Fam}(\bar{\mathbf{B}}) : \mathrm{Fam}(\mathsf{Action}) \to \mathrm{Fam}(\mathsf{SetBundle})$. We show that the action of Fam preserves local fully-faithfullness, but that preservation of local essential surjectivity requires an application of the axiom of choice. Finally, we describe a 2-functor $\Sigma : \mathrm{Fam}(\mathsf{SetBundle}) \to \mathsf{SetBundle}$ performing *summation* of families of set bundles, implicitly using the universal property of the Fam-construction as a free coproduct completion.

Theorem. *The composite 2-functor*

$$\mathbf{B}^* : \mathsf{ActCont} = \mathrm{Fam}(\mathsf{Action}) \xrightarrow{\mathrm{Fam}(\bar{\mathbf{B}})} \mathrm{Fam}(\mathsf{SetBundle}) \xrightarrow{\Sigma} \mathsf{SetBundle} = \mathsf{SymmCont}$$

is locally fully faithful. It takes $(S \triangleright P \triangleleft^\sigma G)$ to a symmetric container with shapes $\sum_s \mathbf{B}G_s$.

This exhibits morphisms of action containers as a well-behaved class of morphisms of symmetric containers: local fully-faithfullness asserts that conjugators of action container morphisms represent exactly identifications of symmetric container morphisms. At the same time, this sheds light on the various ways in which symmetric containers are obtained in practice. For example, [7, Ex. 3.1.2] obtains numerous containers by starting with a group G, and defining $\mathbf{B}^*(1 \triangleright G \triangleleft^\mu G)$ where μ is G acting on itself by multiplication. Similarly, $\mathbf{B}^*(\mathsf{Cyc})$ has shapes $\mathbb{N} \times \mathbf{B}\mathbb{Z} = \mathbb{N} \times S^1$, and as positions n-fold covers of the circle S^1.

Action containers model non-inductive strictly-positive data types. We are interested to see if the same applies to inductive or coinductive types. In the setting of intensional type theory, Ahrens et al. [3] have shown that extensions of general containers (without assumptions on the truncation-level of shapes or positions) admit largest fixed points. Using `Cubical Agda`, Damato et al. [6] show that such containers preserve both smallest and largest fixed points. We want to know: Given an action container F, are smallest and largest fixed points of its extension $[\![\mathbf{B}^* F]\!] : \mathsf{hGpd} \to \mathsf{hGpd}$ in groupoids represented by action containers? I.e. is there a container μF such that $[\![\mathbf{B}^*(\mu F)]\!] = \mu[\![\mathbf{B}^* F]\!]$? If so, is it an initial object in a suitable 2-category of algebras? Does the same apply to the largest fixed point $\nu[\![\mathbf{B}^* F]\!]$? In this case, action containers would give a syntactic description of a sizeable class of (co)inductive data types with symmetries.

We formalize our results using the Agda proof assistant, building on top of the `agda/cubical` library [10]. Our code is freely available at https://github.com/phijor/cubical-containers.

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