Recent progress in the theory of effective Kan fibrations in simplicial sets

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Contents of this talk

The contents of this talk are mostly based on the preprint *Examples and cofibrant generation of effective Kan fibration in simplicial sets*, arXiv2402.10568, written together with Freek Geerligs.

The other main source is the book written together with Eric Faber: *Effective Kan fibration in simplicial sets*, Springer, 2022.

Starting point is Voevodsky's construction of a model of HoTT/UF in simplicial sets.

The simplicial sets model

Theorem (Voevodsky)

The category of simplicial sets carries a model of type theory in which the univalence axiom holds.

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Theorem (Kan-Quillen)

The category of simplicial sets carries a model structure.

Voevodsky uses the fibrations of this model structure, the *Kan fibrations*, to interpret the dependent types.

Voevodsky's proof uses both classical logic and choice (as does the traditional proof of the existence of the Kan-Quillen model structure).

Can this be avoided?

Simplicial sets: the basics

Simplicial sets are presheaves on the simplex category Δ. This category has as:

Objects: For each natural number n the set

$$[n] = \{0, 1, \ldots, n\},$$

thought of as non-empty linearly ordered set.

Morphisms: Monotone maps.

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For each $n \in \mathbb{N}, 0 \leq i \leq n$, we have a *face map*

$$d_i : [n] \to [n+1],$$

the injective map which omits i, and a *degeneracy map*

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Every map in Δ can be written as a composition of face and degeneracy maps.

Simplicial sets: the basics, continued

- We write
$$\Delta^n := y[n]$$
for the representable presheaf on $[n]$ and we refer to Δ^n as the *n-simplex*.
- Subobjects of Δ^n (also known as *sieves*) correspond to subcomplexes of the n-simplex.
- Among the sieves are the *horns* which miss the interior and one face. The horn on Δ^n which misses the face opposite the kth vertex is written Λ^n_k.

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Definition (cofibrant sieve)

A sieve S will be called *cofibrant* if it is pointwise decidable: so if $S \subseteq \Delta^n$, we can decide for any map $\alpha : [m] \to [n]$ whether it belongs to S or not.

In a constructive context it is the cofibrant sieves which really correspond to subcomplexes.

Kan fibrations

The fibrations in the Kan-Quillen model structure are the *Kan fibrations*.

Definition (Kan fibration)

A map $f : Y \to X$ of simplicial sets is a *Kan fibration* if every solid commutative square

$$\begin{array}{ccc} \Lambda^n_k & \longrightarrow & Y \\ \downarrow & \nearrow & \downarrow{\scriptstyle f} \\ \Delta^n & \longrightarrow & X \end{array}$$

has a (not necessarily unique) dotted filler as shown. The object Y is a *Kan complex* if $Y \to 1$ is a Kan fibration.

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In classical maths being a Kan fibration is understood as a *property*. However, let us say that a map $f : Y \to X$ is a *structured Kan fibration* if it comes equipped with an explicit choice of lifts for any commutative square as the one above.

What is your constructive problem?

Recall that the Kan fibrations interpret the dependent types, while the Kan complexes interpret the types.

Theorem (Bezem-Coquand-Parmann)

The classical result which says that A^B is a Kan complex whenever A and B are, does not hold constructively.

I refer to this as the *BCP-obstruction*.

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Note that:

- The classical proof says that A^B is Kan whenever A is.
- We also cannot constructively shown that A^B is a structured Kan complex whenever A and B are. (Parmann)

What should we do now?

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I am aware of three possible responses:

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2. Bite the bullet!
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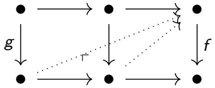
3. This is what I have been trying to do with the theory of effective Kan fibrations. Precursor: the *uniform Kan fibrations* defined by Gambino and Sattler.

Maps structured Kan fibrations lift against

- If f lifts against g, then f also lifts against any pushout of g.



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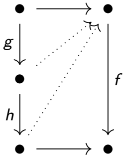

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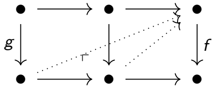

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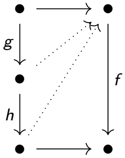

A sequence $S_0 \subseteq S_1 \subseteq \ldots \subseteq S_k$ of cofibrant sieves on Δ^n where each $S_i \subseteq S_{i+1}$ is a pushout of a horn inclusion will be called a *horn pushout sequence*.

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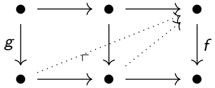

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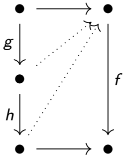

A sequence $S_0 \subseteq S_1 \subseteq \ldots \subseteq S_k$ of cofibrant sieves on Δ^n where each $S_i \subseteq S_{i+1}$ is a pushout of a horn inclusion will be called a *horn pushout sequence*. We conclude: *each structured Kan fibration has induced lifts against inclusions $S \subseteq T$ of cofibrant sieves if S and T are given as the endpoints of a horn pushout sequence.*

Effective Kan fibrations

What happens if we pull back a horn inclusion along a degeneracy?

$$\begin{array}{ccc} S & \longrightarrow & \Lambda^n_k \\ \big\downarrow & \lrcorner & \big\downarrow \\ \Delta^{n+1} & \xrightarrow{s_i} & \Delta^n \end{array}$$

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The inclusion $S \subseteq \Delta^{n+1}$ can be written as the composition of a horn pushout sequence. This decomposition is not unique; however, the induced lift against any structured Kan fibration will be.

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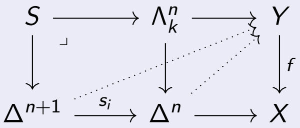
Definition (effective Kan fibration)

A structured Kan fibration $f : Y \to X$ is an *effective Kan fibration* if its induced lifts make any diagram of the following form commute:



Properties of effective Kan fibrations

We have established the following properties of effective Kan fibrations:

Classical correctness: Using classical logic and choice, one can show that every Kan fibration can be equipped with the structure of an effective Kan fibration (jww Eric Faber).

Exponentials: If A is an effective Kan complex, then so is A^B for any simplicial set B. More generally, effective Kan fibrations are closed under Π (jww Eric Faber).

Other type constructors: Effective Kan fibrations interpret the following type constructors: $\Pi, \Sigma, +, \times, 0, 1, \mathbb{N}$. We have a slightly ineffective proof for W as well (jww Shinichiro Tanaka).

Other examples: Simplicial groups (more generally, simplicial Malcev algebras) are Kan (jww Freek Geerligs).

A big open problem

We can construct universes using the Hofmann-Streicher construction. However, I do not know if they are effectively Kan or satisfy univalence.

Double categories

The definition of an effective Kan fibration can be phrased in the language of *double categories*.

Definition (double category)

A double category consists of:

- Objects.
- Horizontal arrows $\bullet \longrightarrow \bullet$ between these objects.
- Vertical arrows $\begin{array}{c}\bullet\\ \downarrow\\ \bullet\end{array}$ between these objects.
- Squares $\begin{array}{ccc}\bullet & \longrightarrow & \bullet\\ \downarrow & & \downarrow\\ \bullet & \longrightarrow & \bullet\end{array}$ which can be composed horizontally and vertically.

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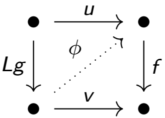
Example

If $\mathcal{C}$ is a category, then there is a double category $\mathrm{Sq}(\mathcal{C})$ whose horizontal and vertical arrows are the morphisms of $\mathcal{C}$, while its squares are the commutative squares in $\mathcal{C}$.

Lifting against double categories

Let $\mathbb{L}$ be a double category and $L : \mathbb{L} \to \mathrm{Sq}(\mathcal{C})$ be a double functor. If $f : Y \to X$ is a morphism in $\mathcal{C}$, then a *right lifting structure against L* is a function which assigns to each vertical morphism g in $\mathbb{L}$ and each square $(u, v) : Lg \to f$ in $\mathrm{Sq}(\mathcal{C})$ a lift $\phi = \phi_g(u, v)$ as shown:



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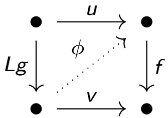
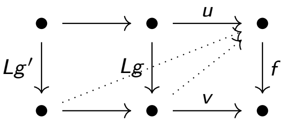
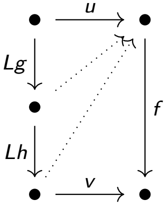


These lifts are required to satisfy two compatibility conditions, a *horizontal* and a *vertical* one, which can be depicted as follows:





A double category for effective Kan fibrations

Let $\mathbb{L}_0$ be the following double category:

- Objects are cofibrant sieves $S \subseteq \Delta^n$.
- Horizontal morphisms are pullback squares $\begin{array}{ccc} S & \longrightarrow & T \\ \downarrow & & \downarrow \\ \Delta^n & \xrightarrow{\alpha} & \Delta^m \end{array}$.
- Vertical morphisms are horn pushout sequences $S_0 \subseteq S_1 \subseteq \ldots \subseteq S_k$.
- A square from $S_0 \subseteq S_1 \subseteq \ldots \subseteq S_k \subseteq \Delta^n$ to $T_0 \subseteq T_1 \subseteq \ldots \subseteq T_l \subseteq \Delta^m$ is given by a map $\alpha : \Delta^n \to \Delta^m$ and a monotone function $f : \{0, \ldots, l\} \to \{0, \ldots, m\}$ such that $f(0) = 0, f(l) = k$ and $\alpha^* T_i = S_{f(i)}$. Such a square is a *face* or *degeneracy square* if α is a face or degeneracy map.

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The double category $\mathbb{L}$ is defined in the same way, but each square is an explicit composition of face and degeneracy squares.

Theorem (Freek Geerligs & BvdB)

A map is an effective Kan fibration iff it has a right lifting structure against the double category $\mathbb{L}$.

Algebraic weak factorisation systems

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Theorem (Bourke & Garner)

If $L : \mathbb{L} \to \mathrm{Sq}(\widehat{\mathcal{C}})$ is a double functor from a small double category, then the maps having the right lifting structure against L are the right maps in an algebraic weak factorisation system (AWFS).

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Ultimately I hope the effective Kan fibrations can be the fibrations in an algebraic model structure (constructively!).

Caveats

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This may sound a bit wacky, but note that the split fibrations interpret the dependent types in the groupoid model of Hoffmann & Streicher. These are not closed under retracts (they are, however, the right class in an AWFS). Of course, the fibrations in the model structure on groupoids are general (not necessarily split) fibrations of groupoids.

If the whole project can be brought to a successful conclusion the picture for simplicial sets will be similar (at least for the constructivist!).

Thank you!