A formal framework for univalent completions

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Introduction

Univalence principle

For mathematical objects:

1. equipped with a notion of "sameness"
2. reasoning invariant

Univalence principle for categories (AKS)

Univalence axiom $\Rightarrow$ univalent categories are necessarily invariant under weak equivalences.

Problem

What about categories with "structure"?

Categories in HoTT/UF

Univalent categories (AKS)

A category X is *univalent* if

$$\mathrm{idtoiso}_{x,y} : (x =_{ob(X)} y) \to (x \cong_X y),$$

is an equivalence of types.

Examples

You know this, hopefully!

However, not every *classical* construction is closed under univalence, such as Kleisli categories constructed via Kleisli morphisms (even if the underlying category is univalent).

Rezk completion

A process to turn a non-univalent category into a univalent one:

Rezk completion (AKS)

The *Rezk completion* of a category: "free univalent" category, i. e. , $\mathrm{Cat}_{\mathrm{univ}} \hookrightarrow \mathrm{Cat}$ has a left 2-adjoint.

Construction (AKS)

Concretely, the Rezk completion of a category is constructed as the full subcategory of *representable* presheaves.

Weak equivalences: sameness for categories (AKS)

Definition

A *weak equivalence* is a functor $f : X \to Y$ such that:

1. f is fully faithful
2. f is essentially surjective

Theorem

If $f : X \to Y$ is a weak equivalence, and Y is univalent, then Y satisfies the universal property of the Rezk completion

Corollary

A category X is univalent if and only for any weak equivalence $f : A \to B$, the precomposition functor $\mathrm{Cat}(f, X)$ is an isomorphism of categories.

Examples

Enriched categories (vdW)

1. (definition) categories equipped with "hom-objects";
2. (weak equivalence) an underlying weak equivalence does not suffice.

Monoidal categories (WMA)

1. (definition) categories equipped with a "monoid structure";
2. (weak equivalence) the underlying functor is a weak equivalence.

Construction of the Rezk completion

The Rezk completion of monoidal, resp. enriched, categories, is constructed as via presheaves.

Goal

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Find a "unifying" theory behind these examples
$\rightsquigarrow$ illuminate and to extend the theory of *univalent categories* in homotopy type theory.

How? $\rightsquigarrow$ apply category theory to itself.

Exercise 9.5 (HoTT book)

How much of this chapter can be done internally to an arbitrary 2-category?

Ingredients

1. univalent objects
2. weak equivalences (eso + ff)
3. Rezk objects

Goal

Determine sufficient (n-dimensional) structure on a bicategory B, to suitably interpret the above ingredients, and their relations.

Yoneda approach (Street, Walters)

Presheaf objects suffice!
$\rightsquigarrow$ Why? Any(_)*thing* is an extension.

Yoneda structure (SW)

A *Yoneda structure* consists of (modulo admissibility):

Structure/Data

A **presheaf object** for $X : \mathrm{B}$ consists of the following structure:

1. an object $\mathbb{P}X$ of presheaves (*presheaf object*);
2. a morphism $よ_X : X \to \mathbb{P}X$ (*Yoneda morphism*).

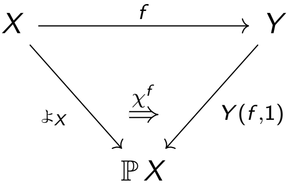
Furthermore, the following statements have a witness:

Axioms/Laws

1. for every $f : X \to Y$, there is given a *left extension* $\big(Y(f,1), \chi^f\big)$ of $よ_X$ along f;
2. each 2-cell χ^f exhibits f as an absolute left lifting of $よ_X$ through $Y(f,1)$.

chi

Let $f : X \to Y$.



Yoneda structure: ExNat precomp

Idea

every morphism is uniquely determined by its action on "generalized objects" and "generalized morphisms" respectively.

The idea is made formal by the following construction, due to Street and Walters.

Construction

Every precomposition functor $\mathrm{B}(f, Z)$ factors through a displayed category over the target (hom-)category, denoted $ExNat(f, Z)$:

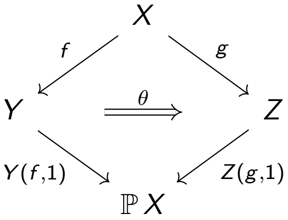
$$\begin{array}{ccc} & & ExNat(f, Z) \\ & \overset{(f \cdot^{e}_{Z} -)}{\nearrow} & \downarrow \pi_1 \\ \mathrm{B}(Y, Z) & \xrightarrow[(f \cdot -)]{} & \mathrm{B}(X, Z) \end{array}$$

Exnat

Let $f : X \to Y$ be a morphism and Z an object. Then

$$ExNat(f, Z) := \int_{g:X\to Z} \mathrm{Mon}(f, g)$$

where $\mathrm{Mon}(f, g)$ consists of (suitable) 2-cells:



Interpretation: univalent objects

Definition

An object Z is *univalent* if $\mathit{ExNat}(f, Z)$ is univalent (as a category) for all $f : X \to Y$.

Proposition

If Z is univalent, then $\hom(X, Z)$ is univalent (for all X).

Conversely:

Proposition

If $\mathbb{P}X$ is hom-wise univalent, then $\mathit{ExNat}(f, Z)$ is displayed univalent over $\mathrm{B}(X, Z)$.
Hence, if Z is hom-wise univalent, then Z is univalent.

Interpretation: fully faithful

Definition (SW)

A morphism $f : X \to Y$ is *fully faithful* if

$$\chi^f : X(1, 1) \Rightarrow Y(f, f),$$

is invertible.

Lemma

f is fully faithful if and only if for all Z,

$$\pi_1 : ExNat(f, Z) \to \mathrm{B}(X, Z),$$

is a (weak) equivalence of categories.

Interpretation: essentially surjective

Definition

A morphism $f : X \to Y$ is *essentially surjective* if for all univalent objects Z,

$$(f \cdot^{e}_{Z} -) : \mathrm{B}(Y, Z) \to ExNat(f, Z),$$

is a weak equivalence of categories.

Remark

If we remove the univalence requirement, we recover those morphisms which are "equivalent on objects".

Conjecture

The essentially surjective morphisms are left orthogonal to those morphisms which are fully faithful and "univalence reflecting/amnestic".

Weak equivalences

Definition

A morphism $f : X \to Y$ is a *weak equivalence* if f is fully faithful and essentially surjective.

Theorem

Let $f : X \to Y$ be a weak equivalence, then:

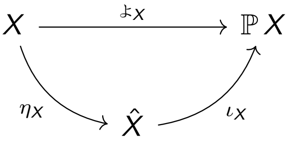
1. *for any univalent object Z,*

$$(f \cdot -) : \mathrm{B}(Y, Z) \to \mathrm{B}(X, Z),$$

   is an isomorphism of categories;
2. *if Y is univalent, then f is a universal arrow for $\mathrm{B}_{\mathrm{univ}} \hookrightarrow \mathrm{B}$;*
3. *if X is univalent, then f is an isomorphism (homwise).*

Rezk completion: construction

Let X be an object. Assume $よ_X : X \to \mathbb{P}X$ factors as:



where

1. $\hat{X}$ is univalent;
2. η_X is essentially surjective;
3. ι_X is fully faithful;

Theorem

If $\mathbb{P}X$ is univalent, then $(\hat{X}, \eta_X)$ is the Rezk completion for X.

Future work

1. admissibility (representability);
2. univalent displayed bicategories (e. g. , pseudomonoids);
3. reflective subbicategories (internal co-completions/exactness);
4. Kleisli objects;
5. internal logic (relative to presheaves);
6. Rezk completion of bicategories.

Conclusion

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Yoneda structures are (almost) sufficient to interpret the "univalent category theory".

Informal conclusion

Presheaves for the win!