A topological reading of (co)inductive definitions in Dependent Type Theories

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Overview

Our work compares the inductive and coinductive methods in Formal Topology with:

- Inductive and Coinductive Definitions of Aczel, "An Introduction to Inductive Definitions";
- W-types and M-types of Martin-Löf's type theory;
- Higher Inductive Types *(and Higher Coinductive Types?)* of Homotopy Type Theory.

Formal Topology

Formal Topology studies topology in a constructive and predicative way by reversing the conceptual order of classical topology:

$$\textit{Points} \rightarrow \textit{Open and Closed subsets} \rightarrow \textit{Basic Opens}$$

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Note: this is not just a matter of taste, but it seems the unavoidable path if one wants to work predicatively.

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Formal Topology

Definition

A *formal topology* consists of a set A of *basic opens*, together with two relations between its elements $a \in A$ and its subsets $V \in \mathcal{P}(A)$:

1. A *basic cover* $a \lhd V$, satisfying:
   - *(reflexivity)* if $a \,\varepsilon\, V$, then $a \lhd V$;
   - *(transitivity)* if $a \lhd U$ and $(\forall x \,\varepsilon\, U) x \lhd V$, then $a \lhd V$.
2. A *positivity relation* $a \ltimes V$, satisfying:
   - *(coreflexivity)* if $a \ltimes V$, then $a \,\varepsilon\, V$;
   - *(cotransitivity)* if $a \ltimes U$ and $(\forall x \in A)(x \ltimes V \Rightarrow x \,\varepsilon\, U)$, then $a \ltimes V$;
   - *(compatibility)* if $a \ltimes V$ and $a \lhd U$, then $(\exists x \,\varepsilon\, V)(x \ltimes U)$.

Spatial intuition: $a \lhd V$ means the basic open a is covered by the union of basic opens in V; $a \ltimes V$ means there exists a point in the basic open a whose basic neighbourhoods are all in V.

(Co)Inductive Methods in Formal Topology

A powerful method to generate a formal topology is by considering a so-called *axiom set*

$$x \in A \vdash I(x) \text{ set} \quad x \in A, y \in I(x) \vdash C(x,y) \in \mathcal{P}(A)$$

We can:

- inductively define the *smallest basic cover* satisfying $a \lhd C(a,i)$ for each $a \in A$, $i \in I(a)$;
- coinductively define the *greatest positivity relation* satisfying $a \ltimes C(a,i)$ for each $a \in A$, $i \in I(a)$.

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Advantages: (co)inductively generated formal topologies behave more nicely; moreover, natural topologies are often (co)inductively generated, e.g. the Baire space, the Cantor space, the line of real numbers.

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Formalisation of (co)inductive methods

The Minimalist Foundation is a dependent type theory introduced to serve:

- as a foundational system for developing Formal Topology with a primitive notion for propositions (as opposed to Martin-Löf's proposition-as-type paradigm);
- as a common core among various foundations of mathematics: its definitions, theorems, and proofs can be exported soundly in any of the most relevant foundations for mathematics.

Maietti and Giovanni Sambin,
"Toward a minimalist foundation for constructive mathematics"

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The (co)inductive methods of Formal Topology have been implemented there as an inductive constructor $\triangleleft$, and a coinductive constructors $\ltimes$. Note: they are both *propositional* constructors (such as $\exists$, $\vee$, etc.)

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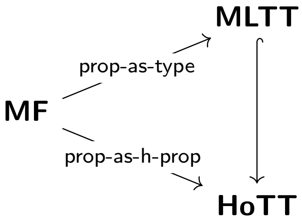
Formalisation of (co)inductive methods

In particular, the Minimalist Foundation is compatible with:

1. Martin-Löf's type theory, by enforcing the proposition-as-type paradigm;
2. Homotopy Type Theory; in particular, propositions are interpreted as h-propositions.



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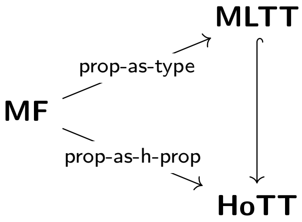

As a consequence, in **HoTT** we get *proof-irrelevant* and *proof-relevant* versions of $\lhd$ and $\ltimes$.

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(Non-)Wellfounded trees

W-types and M-types (and their indexed versions) formalise sets of wellfounded trees and non(-necessarily)-wellfounded trees, respectively.

Gambino and Hyland, "Wellfounded trees and dependent polynomial functors"; van den Berg and De Marchi, "Non-well-founded trees in categories"

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Definition

In a locally cartesian closed category $\mathcal{C}$, a *polynomial endofunctor* is a functor isomorphic to the composite

$$\mathcal{C}/A \xrightarrow{h^*} \mathcal{C}/C \xrightarrow{\Pi_g} \mathcal{C}/B \xrightarrow{\Sigma_f} \mathcal{C}/A$$

for some diagram $A \xleftarrow{f} B \xleftarrow{g} C \xrightarrow{h} A$, called a *polynomial*.

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Definitional semantics

Indexed W-types are defined as initial algebras of polynomial functors. Dually, indexed M-types are defined as terminal coalgebras of polynomial functors.

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The proof-relevant case

Theorem (Maietti, S. 2023)

In **MLTT** + funext, *proof-relevant inductive basic covers are definable using* W*-types, and viceversa.*

Theorem

In **MLTT** + funext, *proof-relevant coinductive positivity relations are definable as dependent* M*-types.*

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Corollary

HoTT *supports the proof-relevant inductive and coinductive methods of Formal Topology.*

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Proof idea

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We show that:

- proof-relevant inductive basic covers are initial algebras of polynomial functors; and proof-relevant coinductive positivity relations are terminal coalgebras of copolynomial functors;
- copolynomial functors are a subclass of polynomial functors.

Aczel's (Co)Inductive definitions

Consider a set A, and a set $\mathcal{R}$ of *inference rules* of the form

$$\frac{P}{a} \qquad \text{with } a \in A \text{ and } P \subset A$$

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Inductive and coinductive definitions formalise the subsets of:

- *Derivable* elements $\mathrm{Ind}_{\mathcal{R}} \subset A$ of the deduction system $\mathcal{R}$
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We defined them in the Minimalist Foundation as propositional constructors.

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Theorem

In the Minimalist Foundation, inductive basic covers and inductive definitions are mutually definable; and so are coinductive positivity relations and coinductive definitions.

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What about proof-irrelevant coinductive positivity relations? We would like to *coinductively define an h-proposition*.

Thierry Coquand and Tosun, "Formal Topology and Univalent Foundations"

Thank you for the attention!

`Agda` formalisation available at
github.com/PietroSabelli/topological-co-induction