Univalent Double Categories

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Overview

Univalence Maxim: Univalent Mathematics ⇔ Category Theory

Provides appropriately weak definitions

Category Theory → Definition in Univalent Mathematics

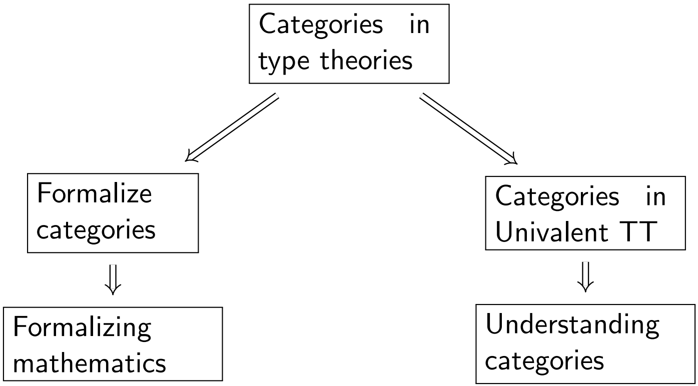
Definition in Univalent Mathematics → Category Theory

Defines categories up to desired equivalence

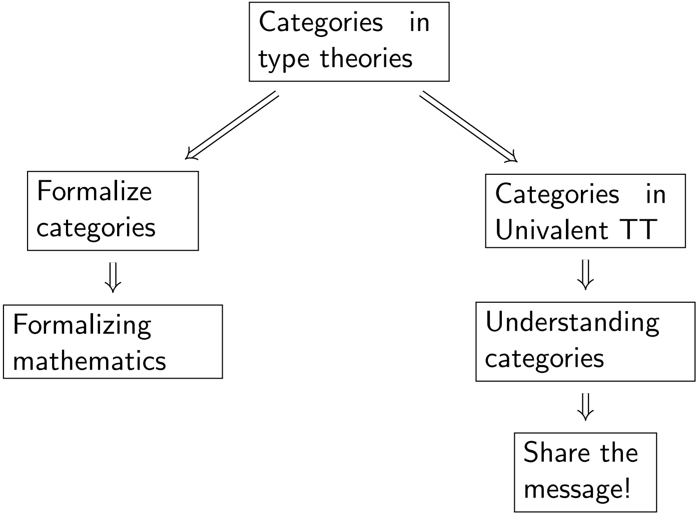
Application: Double categories

Equivalence	**Structure**
Iso. of Double Cat.	Double Setcat.
Iso. of Pseudo Double Cat.	Pseudo Setcat.
Horizontal Equiv. of Double Cat.	Univalent Pseudo Double Cat.
Gregarious Equiv. of Double Cat.	Univalent Weak Double Cat.

Categories in Type Theories



Categories in Type Theories



Formalizing Categories in Type Theories

A category is described as follows:

$$\mathfrak{C}\mathrm{at} := \sum (O : U) \\ (M : O \to O \to U) \\ \left(m : \prod(x\ y\ z : O), (M\ x\ y \to M\ y\ z \to M\ x\ z)\right) \\ \left(e : \prod(x : O), (M\ x\ x)\right), \\ \left(rid : \prod(x\ y : O)(f : M\ x\ y), m\ f\ (e\ x) = f\right) \\ \times \left(lid : \prod(x\ y : O)(g : M\ x\ y), m\ (e\ y)\ g = g\right) \\ \times \left(assoc : \prod(x\ y\ z\ w : O)(f : M\ x\ y)(g : M\ y\ z)(h : M\ z\ w), \\ (m\ (m\ f\ g)h = m\ f\ (m\ g\ h))\right).$$

Formalizing Categories in Univalent Type Theories

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$$
\begin{aligned}
\mathfrak{Cat} := \quad \sum \quad & (O : U) \\
& (M : O \to O \to U) \\
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& \quad (m\ (m\ f\ g) h = m\ f\ (m\ g\ h))\Big) \\
& \times \Big(homsets : \prod (x\ y : O), (isaset\ M\ x\ y)\Big).
\end{aligned}
$$

Power of Univalence

- What are identities in the type $\mathcal{C}\mathrm{at}$?
  Depends on what we focus on:
  - $\mathcal{C}\mathrm{at}_{Set}$: *Setcategories* (type of objects is a set)
    $\Rightarrow$ identities in $\mathcal{C}\mathrm{at}_{Set}$ are isomorphisms.
  - $\mathcal{C}\mathrm{at}_{Univ}$: *Univalent categories* (objects are the isomorphisms).
    $\Rightarrow$ identities in $\mathcal{C}\mathrm{at}_{Univ}$ are equivalences.[1]
- Goes back at least to Rezk:[2]
  - **Nerve:** for $(1,1)$-category of categories vs.
  - **Rezk Nerve:** for $(2,1)$-category of categories (which also has natural isomorphisms).

[1] Benedikt Ahrens, Krzysztof Kapulkin, and Michael Shulman. Univalent categories and the Rezk completion. Math. Structures Comput. Sci 2015

[2] Charles Rezk, A model for the homotopy theory of homotopy theory, Trans. Amer. Math.Soc., 353(2001), no. 3, 973-1007

Both matter

Setcategories and univ. categories do **not** include in each other!

Remark

"Univalent Setcategories"

||

"Univalent categories with at most one iso between two objects"

- **Setcategories:** The set of objects, choosing pullbacks, ...
- **Univalent categories:** Universal properties, ...

2-Categories

A 2-category has:

1. Objects
2. 1-Morphisms between objects
3. Unital and associative composition of 1-morphisms
4. 2-Morphisms between 1-morphisms
5. Unital and associative composition of 2-morphisms

Classifying 2-Categories

In classical literature, 2-categories can be classified up to several layers of strictness (and more!):

(1) **Isomorphism**

(2) **Underlying Equivalence:** Essentially surjective & local isomorphism

(2') **Underlying Equivalence:** Equivalence of underlying category & local isomorphism of 2-morphisms

(3) **Biequivalence:** Essentially surjective & local equivalence at the level of the categories formed by the 1-morphisms and 2-morphisms.

2-Categories in Coq UniMath

Want: 2-categories corresponding to the equivalences!

1. **2-Setcategory:** 2-category with a set of objects and set of 1-morphisms.
2. **Univalent 2-Category:** 2-category with underlying univalent 1-category.
3. **Univalent Bicategory:** Univalent hom categories and objects are equivalences.[3]

[3] Benedikt Ahrens, Dan Frumin, Marco Maggesi, Niccolò Veltri, and Niels van der Weide. Bicategories in univalent foundations. Math. Structures Comput. Sci. 2021

Danger! Danger!

Fact

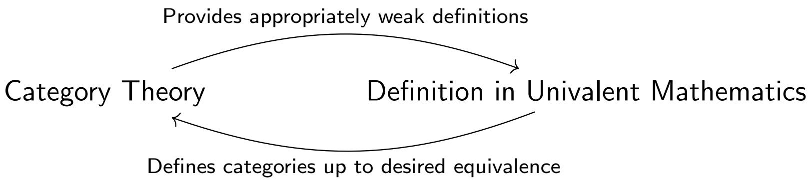
- To capture biequivalences, 1-morphisms need to be the type of isomorphisms of 2-morphisms: generally a 1-type!
- Hence, there is **no underlying category** with associative and unital composition.
- $\Rightarrow$ We need bicategories!

Univalence Maxim

Conclusion

Formalizing and studying categories in univalent foundations (i.e. Coq UniMath) enables precise definitions with built-in equivalences.

This requires choosing appropriately weak definitions!



Univalence Maxim

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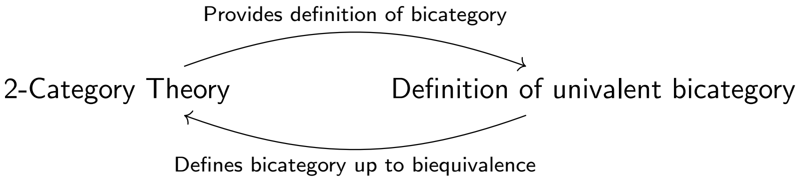


Have a Break! Have a Kit Kat!

Questions?

Comments?

Suggestions?

Enter Double Categories

Some objects come with 2 well-established notions of morphisms:

Object	Morphism 1	Morphism 2	Relevance
Sets	Functions	Relations	Set Theory
Rings	Homomorphisms	Modules	Algebra
Categories	Functors	Pro-functors	Category Theory
Categories	Functors	Lenses[4]	Appl. Cat. Theory
Finite Sets	Functions	Petri Nets[5]	Appl. Cat. Theory
Sets	Functions	Descriptions[6]	Computer Science

We hence generalize categories to a framework that incorporates 2 notions of morphisms that interact well: **double categories**.

4 Bryce Clarke. The double category of lenses. 2023.

5 John C. Baez, Kenny Courser, and Christina Vasilakopoulou. Structured versus decorated cospans. Compositionality, 4(3):39, 2022.

6 Pierre-Évariste Dagand and Conor McBride. A categorical treatment of ornaments. (LICS 2013)

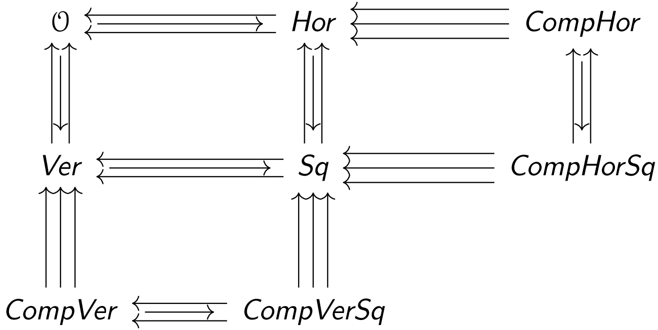
What are Double Categories?

- **Idea:** A category with two types of morphisms.
- **Precise Notion:** A category object in the category of categories.
- **Explicit precise notion:**
  - Objects
  - Horizontal morphisms depending on objects
  - Vertical morphisms depending on objects
  - Squares depending on horizontal/vertical morphisms
  - Identities for objects, horizontal and vertical morphisms
  - Compositions of horizontal, vertical morphisms, squares

  - Unitality of compositions
  - Associativity of compositions

Double Categorical Framework

Basic data and dependency of double categories:



We are missing unitality and associativity, and needed strictness.

Equivalences of Double Categories

Double categories come with various notions of equivalences:

1. **Isomorphisms**
2. **Vertical (horizontal) equivalences:**
   1. Equivalence on the category of objects and vertical (horizontal) morphisms,
   2. Equivalence on the category of horizontal (vertical) morphisms and squares.
3. **Gregarious equivalences:**
   1. Essentially surjective,
   2. Full on horizontal and vertical morphisms up to isomorphism,
   3. Bijection of squares.

Want: Notions of suitably weak double categories fitting all these types of equivalences that we can formalize!

Defining Double Categories for Isomorphisms

This one is as straightforward as the ones before:

Definition

A *strict double setcategory* satisfies the following:

- The types of objects, horizontal morphisms and vertical morphisms are sets
- All associativity and unitality conditions are given strictly

Proposition (Ahrens–North–R.–van der Weide)

The identities of strict double setcategories correspond to isomorphisms.

Defining Double Categories for Vertical Equivalences

- Objects and vertical morphisms should form a univalent category.
- Horizontal morphisms and squares should be a univalent category.
- Horizontal morphisms form a 1-type, so its composition has to be non-strict in general. Here we need associators and unitors.

$\Rightarrow$ We have to weaken the structure in one direction! Such definition exists under the name "pseudo double category" by Grandis[7], Johnson–Yau[8], ...

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[7] Marco Grandis. Higher dimensional categories. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2020. From double to multiple categories.

[8] Niles Johnson and Donald Yau. 2-dimensional categories. Oxford University Press, Oxford, 2021.

Univalent Double Categories for Vertical Equivalences

Building on this literature, we get the following:

Proposition (Ahrens–North–R.–van der Weide)

Identities in the universe of univalent pseudo double categories correspond to vertical equivalences.

Of course we have similar results if we use horizontal equivalences.

Remark

We similarly have a notion of *pseudo double setcategory* with identities isomorphisms of **pseudo double categories**.

Gregarious Univalence

- Gregarious equivalences are symmetric.
- $\Rightarrow$ Need to weaken both horizontal and vertical composition.
- Results in **doubly weak double category** or **weak double category**.

Main Challenge

Weak double categories are not defined in the categorical literature!

Weak Double Categories via Verity Double Bicategories

- Verity defines a notion of **double bicategory**[9].
- Plays the role of "weak pre-double category" or "flagged weak double category"
- Comes with an appropriate univalence condition.[10]
- Can be appropriately adjusted to weak double categories!

Theorem (Ahrens–North–R.–van der Weide)

Identities in the type of univalent weak double categories correspond to gregarious equivalences.

[9] Dominic Verity. Enriched categories, internal categories and change of base. Repr. Theory Appl. Categ., (20):1–266, 2011.

[10] Benedikt Ahrens, Paige Randall North, Michael Shulman and Dimitris Tsementzis. The Univalence Principle, arXiv:2102.06275, to appear in Memoirs of the AMS

A Zoo of Definitions

1. Strict double setcategories
2. Pseudo double setcategories
3. Weak double setcategories
4. Univalent pseudo double categories
5. Univalent weak double categories

In a classical setting we have embeddings:

Strict double cat. $\hookrightarrow$ Pseudo double cat. $\hookrightarrow$ Weak double cat.

This fails univalently (think univalent 2-cat. vs. univalent bicat.).

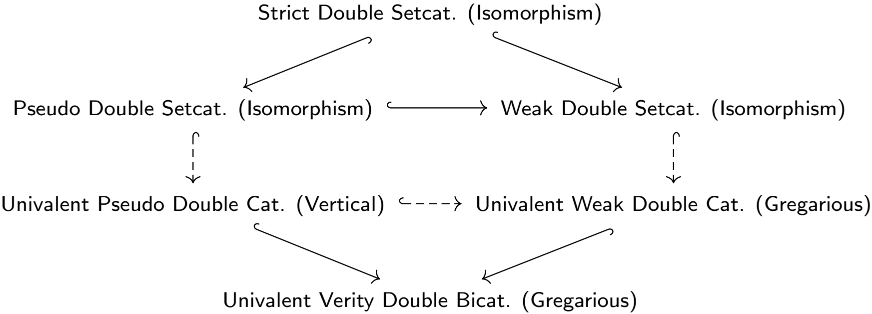
Fact (Fun fact)

The pre-double categorical nature of Verity double bicategories helps us relate definitions in a univalent setting.

Summary of the Double Categorical Picture

↪ : Univalent embedding

⇢ : Only classical embedding



What else is there?

1. **Theory:** Ideally want to lift universes (of univalent weak categories or univalent Verity double bicategories) to categories. Requires tricategories!
2. **Applications:** Do formal category theory in HoTT via *Prof* (categories, functors, profunctors)!

Back to the Overview

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The End!

Thank you!

- Formalization:
  https://github.com/UniMath/UniMath/tree/master/UniMath/Bicategories/DoubleCategories
- Questions:
  1. Ask me in person!
  2. Email: rasekh@mpim-bonn.mpg.de
  3. Check out our papers:
     - *Insights From Univalent Foundations: A Case Study Using Double Categories*
       arXiv:2402.05265
     - *Univalent Double Categories*
       arXiv:2310.09220, doi:10.1145/3636501.3636955