Differential Geometry in Synthetic Algebraic Geometry

Hugo Moeneclaey
j.w.w Felix Cherubini, Matthias Hutzler and David Wärn

HoTT-UF 2024
Leuven

Overview

Goal

Import differential geometry tools to synthetic algebraic geometry.

Draft

`https://felix-cherubini.de/diffgeo.pdf`

Overview

Goal

Import differential geometry tools to synthetic algebraic geometry.

Draft

`https://felix-cherubini.de/diffgeo.pdf`

Today

Focus on smoothness for affine schemes.
Give examples of synthetic proofs.

Overview

Goal

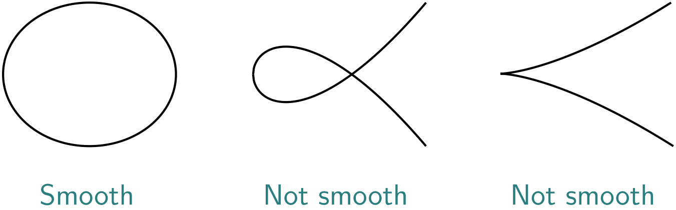
Import differential geometry tools to synthetic algebraic geometry.

Draft

`https://felix-cherubini.de/diffgeo.pdf`

Today

Focus on smoothness for affine schemes.
Give examples of synthetic proofs.



Outline

Synthetic algebraic geometry

Smoothness for arbitrary types

Smoothness for affine schemes

Outline

Synthetic algebraic geometry

Smoothness for arbitrary types

Smoothness for affine schemes

What is synthetic algebraic geometry?

It consists of HoTT plus 3 axioms:

What is synthetic algebraic geometry?

It consists of HoTT plus 3 axioms:

Axiom 1

There is a local ring R.

What is synthetic algebraic geometry?

It consists of HoTT plus 3 axioms:

Axiom 1

There is a local ring R.

R is a set.

Affine schemes

For A a finitely presented algebra, we define:

$$Spec(A) = Hom_{Alg}(A, R)$$

Affine schemes

For A a finitely presented algebra, we define:

$$Spec(A) = Hom_{Alg}(A, R)$$

Example

If:

$$A = R[X]/P$$

then:

$$Spec(A) = \{x : R \mid P(x) = 0\}$$

Affine schemes

For A a finitely presented algebra, we define:

$$Spec(A) = Hom_{Alg}(A, R)$$

Example

If:

$$A = R[X]/P$$

then:

$$Spec(A) = \{x : R \mid P(x) = 0\}$$

Definition

A type X is an affine scheme if there is an f.p. algebra A such that:

$$X = Spec(A)$$

Axiom 2: Duality

For any f.p. algebra A the map:

$$A \to R^{Spec(A)}$$

is an equivalence.

Axiom 2: Duality

For any f.p. algebra A the map:

$$A \to R^{Spec(A)}$$

is an equivalence.

Then:

- $Spec : \{f.p.\ algebras\} \simeq \{Affine\ schemes\}$

Axiom 2: Duality

For any f.p. algebra A the map:

$$A \to R^{Spec(A)}$$

is an equivalence.

Then:

- $Spec : \{f.p.\ algebras\} \simeq \{Affine\ schemes\}$
- All maps between affine schemes are polynomials.

Axiom 3: Zariski local choice

Affine schemes enjoys a weakening of the axiom of choice.

Outline

Synthetic algebraic geometry

Smoothness for arbitrary types

Smoothness for affine schemes

Closed propositions

Definition

A proposition P is closed if there exist $r_1, \ldots, r_n : R$ such that:

$$P \leftrightarrow (r_1 = 0 \wedge \ldots \wedge r_n = 0)$$

Closed propositions

Definition

A proposition P is closed if there exist $r_1, \ldots, r_n : R$ such that:

$$P \leftrightarrow (r_1 = 0 \wedge \ldots \wedge r_n = 0)$$

Lemma

Let P be a closed proposition, TFAE:

(1) There exist $r_1, \ldots, r_n : R$ nilpotent such that:

$$P \leftrightarrow (r_1 = 0 \wedge \ldots \wedge r_n = 0)$$

(2) $\neg\neg P$.

Such a proposition is called closed dense.

Smoothness

Definition

A type X is smooth if for all closed dense proposition P the map:

$$X \to X^P$$

is surjective.

Smoothness

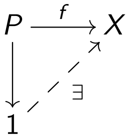
Definition

A type X is smooth if for all closed dense proposition P the map:

$$X \to X^P$$

is surjective.

This means that for all $f : P \to X$ we can fill:



Smoothness

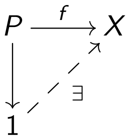
Definition

A type X is smooth if for all closed dense proposition P the map:

$$X \to X^P$$

is surjective.

This means that for all $f : P \to X$ we can fill:



What does this has to do with smoothness?

Outline

Synthetic algebraic geometry

Smoothness for arbitrary types

Smoothness for affine schemes

Examples

Example 1

The affine scheme:

$$R = Spec(R[X])$$

is smooth.

Examples

Example 1

The affine scheme:

$$R = Spec(R[X])$$

is smooth.

We have to merely find a lift in:

$$\begin{array}{ccc} r_1 = 0 \wedge \ldots \wedge r_n = 0 & \longrightarrow & R \\ \downarrow & \overset{?}{\nearrow} & \\ 1 & & \end{array}$$

Examples

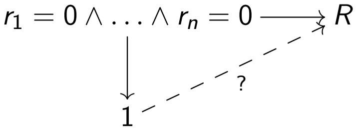
Example 1

The affine scheme:

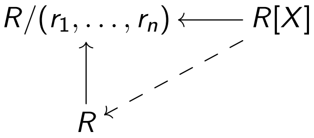
$$R = Spec(R[X])$$

is smooth.

We have to merely find a lift in:



By duality it is enough to merely find a lift in:



Example 2

The affine scheme:

$$Spec(R[X, Y]/(XY)) = \{x, y : R \mid xy = 0\}$$

is not smooth.

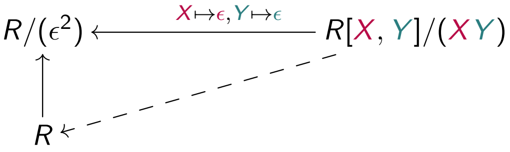
Example 2

The affine scheme:

$$Spec(R[X, Y]/(XY)) = \{x, y : R \mid xy = 0\}$$

is not smooth.

If it was smooth, for all $\epsilon : R$ such that $\epsilon^3 = 0$ the lift in:



gives $r, r' : (\epsilon^2)$ such that $(\epsilon + r)(\epsilon + r') = 0$, so that $\epsilon^2 = 0$.

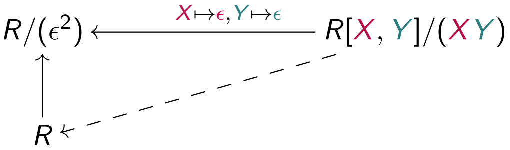
Example 2

The affine scheme:

$$Spec(R[X, Y]/(XY)) = \{x, y : R \mid xy = 0\}$$

is not smooth.

If it was smooth, for all $\epsilon : R$ such that $\epsilon^3 = 0$ the lift in:



gives $r, r' : (\epsilon^2)$ such that $(\epsilon + r)(\epsilon + r') = 0$, so that $\epsilon^2 = 0$.
Then:

$$\{x : R \mid x^2 = 0\} = \{x : R \mid x^3 = 0\}$$

which by duality implies:

$$R[X]/(X^2) = R[X]/(X^3)$$

Tangent spaces

We write:

$$\mathbb{D} := Spec(R[X]/X^2) = \{x : R \mid x^2 = 0\}$$

Tangent spaces

We write:

$$\mathbb{D} := Spec(R[X]/X^2) = \{x : R \mid x^2 = 0\}$$

Definition

For X a type and $p : X$, the tangent space of X at p is:

$$T_p(X) := \{t : \mathbb{D} \to X \mid t(0) = p\}$$

Tangent spaces

We write:

$$\mathbb{D} := Spec(R[X]/X^2) = \{x : R \mid x^2 = 0\}$$

Definition

For X a type and $p : X$, the tangent space of X at p is:

$$T_p(X) := \{t : \mathbb{D} \to X \mid t(0) = p\}$$

We have the tangent bundle:

$$ev_0 : X^{\mathbb{D}} \to X$$

Tangent spaces

We write:

$$\mathbb{D} := Spec(R[X]/X^2) = \{x : R \mid x^2 = 0\}$$

Definition

For X a type and $p : X$, the tangent space of X at p is:

$$T_p(X) := \{t : \mathbb{D} \to X \mid t(0) = p\}$$

We have the tangent bundle:

$$ev_0 : X^{\mathbb{D}} \to X$$

For any map $f : X \to Y$ and $p : X$ we have the differential:

$$df_p : T_p(X) \to T_{f(p)}(Y)$$

Maps with smooth fibers

Proposition

Let $f : X \to Y$ be a map between affine schemes with X smooth. TFAE:

- For all $p : X$ the differential df_p is surjective.
- The fibers of f are smooth.

Tangent spaces of smooth affine schemes

A module M is:

- Finite free if there is $k : \mathbb{N}$ such that $M = R^k$.
- Finitely copresented if it is the kernel of a map $R^m \to R^n$.

Tangent spaces of smooth affine schemes

A module M is:

- Finite free if there is $k : \mathbb{N}$ such that $M = R^k$.
- Finitely copresented if it is the kernel of a map $R^m \to R^n$.

Proposition

Let X be a smooth affine scheme with $p : X$.
Then $T_p(X)$ is finite free.

Tangent spaces of smooth affine schemes

A module M is:

- Finite free if there is $k : \mathbb{N}$ such that $M = R^k$.
- Finitely copresented if it is the kernel of a map $R^m \to R^n$.

Proposition

Let X be a smooth affine scheme with $p : X$.
Then $T_p(X)$ is finite free.

General idea:

Tangent spaces of smooth affine schemes

A module M is:

- Finite free if there is $k : \mathbb{N}$ such that $M = R^k$.
- Finitely copresented if it is the kernel of a map $R^m \to R^n$.

Proposition

Let X be a smooth affine scheme with $p : X$.
Then $T_p(X)$ is finite free.

General idea:

1. Tangent spaces of affine schemes are finitely copresented.

Tangent spaces of smooth affine schemes

A module M is:

- Finite free if there is $k : \mathbb{N}$ such that $M = R^k$.
- Finitely copresented if it is the kernel of a map $R^m \to R^n$.

Proposition

Let X be a smooth affine scheme with $p : X$.
Then $T_p(X)$ is finite free.

General idea:

1. Tangent spaces of affine schemes are finitely copresented.
2. Tangent space of smooth affine schemes are smooth.

Tangent spaces of smooth affine schemes

A module M is:

- Finite free if there is $k : \mathbb{N}$ such that $M = R^k$.
- Finitely copresented if it is the kernel of a map $R^m \to R^n$.

Proposition

Let X be a smooth affine scheme with $p : X$.
Then $T_p(X)$ is finite free.

General idea:

1. Tangent spaces of affine schemes are finitely copresented.
2. Tangent space of smooth affine schemes are smooth.
3. Smooth finitely copresented modules are finite free.

Lemma 2

Let X be a smooth affine scheme with $p : X$.
Then $T_p(X)$ is smooth.

Lemma 2

Let X be a smooth affine scheme with $p : X$.
Then $T_p(X)$ is smooth.

We have to show that the tangent bundle:

$$X^{\mathbb{D}} \to X$$

has smooth fibers.

Lemma 2

Let X be a smooth affine scheme with $p : X$.
Then $T_p(X)$ is smooth.

We have to show that the tangent bundle:

$$X^{\mathbb{D}} \to X$$

has smooth fibers.

- $X^{\mathbb{D}}$ is an affine scheme as a dependent sum of affine schemes.

Lemma 2

Let X be a smooth affine scheme with $p : X$.
Then $T_p(X)$ is smooth.

We have to show that the tangent bundle:

$$X^{\mathbb{D}} \to X$$

has smooth fibers.

- $X^{\mathbb{D}}$ is an affine scheme as a dependent sum of affine schemes.
- $X^{\mathbb{D}}$ is smooth as X is smooth and $\mathbb{D}$ has choice.

Lemma 2

Let X be a smooth affine scheme with $p : X$.
Then $T_p(X)$ is smooth.

We have to show that the tangent bundle:

$$X^{\mathbb{D}} \to X$$

has smooth fibers.

- $X^{\mathbb{D}}$ is an affine scheme as a dependent sum of affine schemes.
- $X^{\mathbb{D}}$ is smooth as X is smooth and $\mathbb{D}$ has choice.
- We need to check its differentials are surjective.

We need to merely find a lift in:

$$\begin{array}{ccc} 1 & \longrightarrow & X^{\mathbb{D}} \\ \downarrow & \nearrow & \downarrow \\ \mathbb{D} & \longrightarrow & X \end{array}$$

We need to merely find a lift in:

$$\begin{array}{ccc} 1 & \longrightarrow & X^{\mathbb{D}} \\ \downarrow & \nearrow & \downarrow \\ \mathbb{D} & \longrightarrow & X \end{array}$$

or equivalently in:

$$\begin{array}{ccc} \mathbb{D} \times 1 \coprod_{1 \times 1} \mathbb{D} \times 1 & \longrightarrow & X \\ \downarrow & \nearrow & \\ \mathbb{D} \times \mathbb{D} & & \end{array}$$

We need to merely find a lift in:

$$\begin{array}{ccc} 1 & \longrightarrow & X^{\mathbb{D}} \\ \downarrow & \nearrow & \downarrow \\ \mathbb{D} & \longrightarrow & X \end{array}$$

or equivalently in:

$$\begin{array}{ccc} \mathbb{D} \times 1 \coprod_{1 \times 1} \mathbb{D} \times 1 & \longrightarrow & X \\ \downarrow & \nearrow & \\ \mathbb{D} \times \mathbb{D} & & \end{array}$$

But $\mathbb{D} \times \mathbb{D}$ has choice so it is enough that for all $(\epsilon, \delta) : \mathbb{D} \times \mathbb{D}$ we merely find a lift in:

$$\begin{array}{ccc} \epsilon = 0 \vee \delta = 0 & \longrightarrow & X \\ \downarrow & \nearrow & \\ 1 & & \end{array}$$

We need to merely find a lift in:

$$\begin{array}{ccc} 1 & \longrightarrow & X^{\mathbb{D}} \\ \downarrow & \nearrow & \downarrow \\ \mathbb{D} & \longrightarrow & X \end{array} \qquad \text{or equivalently in:} \qquad \begin{array}{ccc} \mathbb{D} \times 1 \coprod_{1 \times 1} \mathbb{D} \times 1 & \longrightarrow & X \\ \downarrow & \nearrow & \\ \mathbb{D} \times \mathbb{D} & & \end{array}$$

But $\mathbb{D} \times \mathbb{D}$ has choice so it is enough that for all $(\epsilon, \delta) : \mathbb{D} \times \mathbb{D}$ we merely find a lift in:

$$\begin{array}{ccc} \epsilon = 0 \vee \delta = 0 & \longrightarrow & X \\ \downarrow & \nearrow & \\ 1 & & \end{array}$$

But we can do this as X is smooth.

Lemma

Let P be a closed proposition, if:

$$\neg\neg P \to P$$

then we have that:

$$P \vee \neg P$$

Lemma

Let P be a closed proposition, if:

$$\neg\neg P \to P$$

then we have that:

$$P \vee \neg P$$

This is an application of Nakayama lemma in the local ring R.

Lemma

Let $M : R^m \to R^n$ be a linear map with smooth kernel. Then:

$$M = 0 \vee M \neq 0$$

Lemma

Let $M : R^m \to R^n$ be a linear map with smooth kernel. Then:

$$M = 0 \vee M \neq 0$$

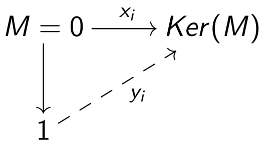
It is enough to check $\neg\neg(M = 0) \to M = 0$.

Lemma

Let $M : R^m \to R^n$ be a linear map with smooth kernel. Then:

$$M = 0 \vee M \neq 0$$

It is enough to check $\neg\neg(M = 0) \to M = 0$.
Assume $\neg\neg(M = 0)$. Take (x_i) a basis of R^m, we have lifts in:

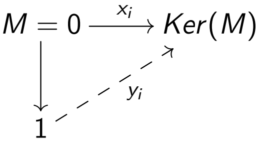


Lemma

Let $M : R^m \to R^n$ be a linear map with smooth kernel. Then:

$$M = 0 \vee M \neq 0$$

It is enough to check $\neg\neg(M = 0) \to M = 0$.
Assume $\neg\neg(M = 0)$. Take (x_i) a basis of R^m, we have lifts in:



But:

- If $M = 0$ then (y_i) is equal to (x_i) so it is a basis of R^m.
- We have $\neg\neg(M = 0)$.
- Being a basis is $\neg\neg$-stable.

So (y_i) is a basis of R^m and $Ker(M) = R^m$ so $M = 0$.

Lemma 3

Let $M : R^m \to R^n$ be a linear map with smooth kernel. Then $Ker(M)$ is finite free.

Lemma 3

Let $M : R^m \to R^n$ be a linear map with smooth kernel. Then $Ker(M)$ is finite free.

By induction on m. Apply the previous lemma.

Lemma 3

Let $M : R^m \to R^n$ be a linear map with smooth kernel. Then $Ker(M)$ is finite free.

By induction on m. Apply the previous lemma.

- If $M = 0$, then $Ker(M) = R^m$ and it is finite free.

Lemma 3

Let $M : R^m \to R^n$ be a linear map with smooth kernel. Then $Ker(M)$ is finite free.

By induction on m. Apply the previous lemma.

- If $M = 0$, then $Ker(M) = R^m$ and it is finite free.
- If $M \neq 0$, then M has an invertible coefficient. By Gaussian elimination we get a linear map $N : R^{m-1} \to R^{n-1}$ with the same kernel.

Conclusion

Today:

- Showcased a couple of synthetic proofs.
- Gave some nice properties of smoothness for affine schemes.

In the notes:

- Justify smoothness through its connections with étaleness.
- Prove smoothness for general types is well-behaved.
- Give an explicit Zariski local description of smooth schemes.
- And much more!

Appendix: Explicit Zariski local description

Definition

A standard smooth scheme is an affine scheme of the form:

$$Spec\Big(\big(R[X_1,\ldots,X_n,Y_1,\ldots,Y_k]/(P_1,\ldots,P_n)\big)_G\Big)$$

where $Jac(P_1,\ldots,P_n) \mid G$.

Theorem

Let X be a scheme, TFAE:

- X is smooth.
- X has a finite open cover by standard smooth schemes.

Appendix: Smoothness is well behaved

Lemma

Open propositions are smooth.

Lemma

Smooth types are closed by Σ.

Lemma

If D has choice and X is smooth, then X^D is smooth.

Lemma

The image of a smooth type by any map is smooth.

Lemma

A type X is smooth if and only is $||X||_0$ is smooth.