Connected Covers in Cubical Agda

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1. Introduction and Motivation

Connected covers were first studied by Cartan and Serre [CS52] and Whitehead [Whi52].

For each n, this construction gives us a “universal” n-connected space over a fixed, pointed base.

If X is the base, we denote this space $X\langle n\rangle$.

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The latter two are used in the proof of the Serre finiteness theorem due to Barton and Campion [BC].

2. Connected Covers

Let X be a pointed space.

Recursive definition of $X\langle n\rangle$:

$$\begin{aligned} X\langle -1\rangle &= X \\ X\langle n+1\rangle &= \mathrm{fiber}_{|\cdot|_{n+1}}\left(|\,\mathrm{pt}_{X\langle n\rangle}\,|_{n+1}\right) \end{aligned}$$

With the obvious point.

Diagram:

$$\begin{array}{ccc} X\langle n+1\rangle & \longrightarrow & X\langle n\rangle \\ \downarrow & \lrcorner & \downarrow \\ 1 & \longrightarrow & \|X\langle n\rangle\|_{n+1} \end{array}$$

2. Connected Covers

Alternative definition:

$$X\langle n\rangle = \mathrm{fiber}_{|\cdot|_n}(|\,\mathrm{pt}_X\,|_n)$$

Diagram:

$$\begin{array}{ccc} X\langle n\rangle & \longrightarrow & X \\ \downarrow & \lrcorner & \downarrow \\ 1 & \longrightarrow & \|X\|_n \end{array}$$

As part of the formalization project, there's a proof that these definitions are equivalent. But it's not very illuminating.

2. Connected Covers

$$X\langle n\rangle = \mathrm{fiber}_{|\cdot|}(\mathrm{pt}_X)$$

Alternatively we can take this to be the definition of $X\langle n\rangle$ and then it is possible to show that it satisfies a universal property, and our original formulation follows.

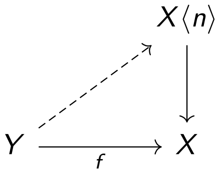
2. Connected Covers

$X\langle n\rangle$ is pointed, n-connected and moreover:

It is the terminal pointed, n-connected space with a map into X

(This is the "universal property" referred to above)

Meaning: if Y is pointed and n-connected, and $f : Y \to X$, then there is a unique filler in the diagram below



(See [CS23; BR23])

2. Connected Covers

It follows that we have group identities

$$\pi_k(X\langle n\rangle) = \begin{cases} 0 & \text{if } k \leq n \\ \pi_k(X) & \text{if } k > n \end{cases}$$

Because the n-sphere is $(n-1)$-connected.

2. Connected Covers

Using either definition, we find a map $X\langle n+1\rangle \to X\langle n\rangle$.

We mentioned before that it is useful to know that the fiber of this map is $K(\pi_{n+1}(X), n)$

The formal proof of this uses 2 extra tools.

Tools

Puppe's Lemma:

If $X \to Y \to Z$ is a fiber sequence, then so is $\Omega Z \to X \to Y$.

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If $X \to Y \to Z$ is a fiber sequence, then so is $\Omega Z \to X \to Y$.

Follows from the pullback lemma:

$$\begin{array}{ccccc} \Omega Z & \longrightarrow & X & \longrightarrow & 1 \\ \downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow \\ 1 & \longrightarrow & Y & \longrightarrow & Z \end{array}$$

Tools

Whitehead's Principle:

If X and Y are n-truncated spaces and $f : X \to Y$ is such that $\|f\|_0$ is a bijection, and $\pi_n(f, x) : \pi_n(X, x) \to \pi_n(Y, f(x))$ is an isomorphism for each $x : X$ and each $n \geq 1$, then f is an equivalence of types.

Tools

Whitehead's Principle:

Follows from the lemma that for any X and Y, if $f : X \to Y$ is such that $\|f\|_0$ is a bijection and $\Omega(f, x) : \Omega(X, x) \to \Omega(Y, f(x))$ is an equivalence of types, then f is an equivalence of types.

```
ΩEquiv→Equiv : {A B : Type ℓ}
  (f : A → B)
  (hf0 : isEquiv (map f))
  (hf : (a : A)
       → isEquiv
          ( fst (Ω→ {A = (A , a)} {B = (B , f a)} (f , refl))))
  → isEquiv f
```

Tools

```
WhiteheadsLemma {n = zero} hA hB f hf0 hf = isEquivFromIsContr f hA hB
WhiteheadsLemma {A = A} {B = B} {n = suc n} hA hB f hf0 hf =
  ΩEquiv→Equiv
  ( f)
  ( hf0)
  ( λ a → WhiteheadsLemma
          ( isOfHLevelPath' n hA a a)
          ( isOfHLevelPath' n hB (f a) (f a))
          ( fst (Ω→ {A = (A , a)} {B = (B , f a)} (f , refl)))
          ( hf a 0)
          ( ΩWhiteheadHyp a))
```

Tools

Corollary:

If X is n-connected and $n+1$-truncated, then:

$$X = K(\pi_{n+1}(X), n+1)$$

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If X is n-connected and $n + 1$-truncated, then:

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In fact, something stronger is true: the map $K(-, n + 1)$ is part of an equivalence between the type of abelian groups and the type of n-connected, $n + 1$-truncated types, and its inverse is π_{n+1} [Doo18].

But we do not use this here.

2. Connected Covers

Back to our goal.

It follows from the corollary to Whitehead's principle just mentioned, and some of the facts about connected covers mentioned above, that

$$\|X\langle n\rangle\|_{n+1} = K(\pi_{n+1}(X), n+1)$$

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So, using Puppe's lemma, we have a fiber sequence:

$$K(\pi_{n+1}(X), n) \to X\langle n+1\rangle \to X\langle n\rangle$$

Conclusion: Code

```
EM<->FibSeq : (X : Pointed ℓ) (n : ℕ)
  → FiberSeq (X < (2 + n) >) (X < (suc n) >) (EM∙ (πAb n X) (2 + n))
EM<->FibSeq X n =
  BaseEqFiberSeq
  ( TruncConnCovEqEM∙ X n)
  ( ConnCovFiberSeq X (suc n))


FibSeqEqEM∙ : (X : Pointed ℓ) (n : ℕ)
  → FiberSeq (Ω (EM∙ (πAb n X) (2 + n))) (X < (2 + n) >) (X < (suc n) >)
  ≡ FiberSeq (EM∙ (πAb n X) (suc n)) (X < (2 + n) >) (X < (suc n) >)
FibSeqEqEM∙ X n i =
  FiberSeq
  ( (EM≃ΩEM+1∙ {G = πAb n X} (suc n)) (~ i))
  ( X < (2 + n) >)
  ( X < (suc n) >)

<->EMFibSeq : (X : Pointed ℓ) (n : ℕ)
  → FiberSeq (EM∙ (πAb n X) (suc n)) (X < (2 + n) >) (X < (suc n) >)
<->EMFibSeq X n = transport (FibSeqEqEM∙ X n) (puppe (EM<->FibSeq X n))

U:**-  EMIsFiber.agda    Bot L63    Git-main   (Agda:Checked)
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References I

[BC] Reid Barton and Tim Campion. *The Finite Presentability of $\pi_k(S^m)$ via Ganea's Theorem*. Unpublished.

[BDR18] Ulrik Buchholtz, Floris van Doorn, and Egbert Rijke. "Higher Groups in Homotopy Type Theory". In: *LICS '18: Proceedings of the 33rd Annual ACM/IEEE Symposium on Logic in Computer Science*. 2018.

[BR23] Ulrik Buchholtz and Egbert Rijke. "The Long Exact Sequence of Homotopy *n*-Groups". In: *Mathematical Structures in Computer Science* 33.8 (2023), pp. 679–687.

[CS23] J. Daniel Christensen and Luis Scoccola. "The Hurewicz Theorem in Homotopy Type Theory". In: *Algebraic & Geometric Topology* 23.5 (2023), pp. 2107–2140.

References II

[CS52] Henri Cartan and Jean-Pierre Serre. “Espaces Fibrés et Groupes d’Homotopie. I. Constructions Générales”. In: *Comptes Redus Hebdomadaires de Séances de l’Académie des Sciences*. 1952, pp. 288–290.

[Doo18] Floris van Doorn. “On the Formalization of Higher Inductive Types and Synthetic Homotopy Theory”. PhD thesis. Carnegie Mellon University, 2018.

[Whi52] George W. Whitehead. “Fiber Spaces and the Eilenberg Homology Groups”. In: *Proceedings of the National Academy of Sciences*. Vol. 38. 5. 1952, pp. 426–430.

Appendix: Sketched Example

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The 1st connected cover ("universal cover") of this space is $\bigvee_{\mathbb{Z}} S^2$

Given this, we can calculate $\pi_2(S^1 \vee S^2)$:

$$\pi_2\left(\bigvee_{\mathbb{Z}} S^2\right) = H_2\left(\bigvee_{\mathbb{Z}} S^2\right) = \bigoplus_{\mathbb{Z}} H_2(S^2) = \bigoplus_{\mathbb{Z}} \mathbb{Z}$$

Appendix: Sketched Example

Why is $\bigvee_{\mathbb{Z}} S^2$ the 1st connected cover?

This argument was pointed out to me by David Wärn after my talk:

First observe:

$\|S^1 \vee S^2\|_1$ is equivalent to S^1 and the truncation map becomes the pointed projection under this equivalence.

Appendix: Sketched Example

The wedge sum is defined using a pushout diagram like so:

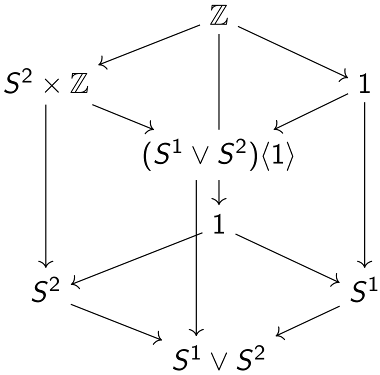
$$\begin{array}{ccc} 1 & \longrightarrow & S^1 \\ \downarrow & & \downarrow \\ S^2 & \longrightarrow & S^1 \vee S^2 \end{array}$$

Taking the fibers with the composites of the map $S^1 \vee S^2 \to S^1$ gives the following diagram:

$$\begin{array}{ccc} \mathbb{Z} & \longrightarrow & 1 \\ \downarrow & & \downarrow \\ S^2 \times \mathbb{Z} & \longrightarrow & (S^1 \vee S^2)\langle 1 \rangle \end{array}$$

Appendix: Sketched Example

Placing our two diagrams one atop the other and applying the pullback lemma, we have that each face of the cube below is a pullback square:



Appendix: Sketched Example

It follows by descent [c.f. e.g. BDR18] that this square:

$$\begin{array}{ccc} \mathbb{Z} & \longrightarrow & 1 \\ \downarrow & & \downarrow \\ S^2 \times \mathbb{Z} & \longrightarrow & S^1 \vee S^2\langle 1 \rangle \end{array}$$

is a pushout square.

So that $(S^1 \vee S^2)\langle 1 \rangle \simeq \bigvee_{\mathbb{Z}} S^2$