The Steenrod Squares in HoTT Revisited

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Introduction

- Let's leave the cellular world
- This talk is about the Steenrod squares, a construction on the usual 'representable' cohomology theory in HoTT.
- Originally investigated by Brunerie [1]. The goal of this project is to finish what he started.

Cohomology in HoTT: a crash course

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  - and, when G is a (comm) ring R , a graded multiplication $\smile : K(R, n) \times K(R, m) \to K(R, n+m)$ called the *cup product*.
- These operations turn $H^*(A, R)$ into a graded commutative ring with addition $+$ and multiplication $\smile$.

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- In this talk, we will revisit Brunerie's definition and present proofs of several of their key properties.

The Steenrod Squares, axiomatically

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Theorem 1 (partially proved)

There is a family of pointed maps $\mathrm{Sq}^n_{(i)} : K_i \to_\star K_{i+n}$ *s.t.*

(a) $\mathrm{Sq}^0(x) = x$

(b) $\mathrm{Sq}^n_n = x^2 \qquad (:= x \smile x)$

(c) $\mathrm{Sq}^n_i(x) = 0$ *if* $n > i$

(d) *The Cartan formula*

$$\mathrm{Sq}^n(x \smile y) = \sum_{m+k=n} \mathrm{Sq}^m(x) \smile \mathrm{Sq}^k(y)$$

The Steenrod Squares, axiomatically

Theorem 1 (ctd.)

Furthermore, Sq^n *respects suspension*

$$\begin{tikzcd} K_i \arrow[r, "\mathrm{Sq}^n"] \arrow[d, "\wr"'] & K_{i+n} \arrow[d, "\wr"] \\ \Omega(K_{i+1}) \arrow[r, "\Omega(\mathrm{Sq}^n)"'] & \Omega(K_{i+1+n}) \end{tikzcd} \qquad (\Sigma\text{-axiom})$$

and the Adem relations: when $i < 2j$*, we have*

$$\mathrm{Sq}^i \circ \mathrm{Sq}^j(x) = \sum_{k=0}^{\lfloor i/2 \rfloor} \binom{j-k-1}{i-2k}_{\bmod 2} \mathrm{Sq}^{i+j-k} \circ \mathrm{Sq}^k(x)$$

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1. A Brunerie style construction
2. A direct construction

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- Idea: obtain Steenrod squares as a special case of an 'unordered cup product'

- Let $X : \mathbb{R}P^\infty$ and $A : X \to \mathcal{U}$. We call the type $\Pi_{x:X} A(x)$ the type of *unordered pairs of A (rel. X)*

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- To get this, we need to endow this function space with more structure. To this end, we'll use joins.

Definition 2

Given $X : \mathbb{R}P^\infty$ and $A : X \to \mathcal{U}$, the **unordered join of** A **(rel. X)** is the pushout

$$\begin{tikzcd}
X \times \Pi_{(x:X)} A(x) \arrow[r] \arrow[d] & \Pi_{(x:X)} A(x) \arrow[d] \\
\Sigma_{x:X} A(x) \arrow[r] & \ast_{x:X} A(x) \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

This coincides with the usual join when X is $\mathbb{2}$

Definition 3

Given $X : \mathbb{R}P^\infty$ and $A : X \to \mathcal{U}_\star$ and $B : \mathcal{U}_\star$, we define the type of bi-homs by

$$A \to_\star^X B := \sum_{F : \Pi_{x:X} A(x) \to B} \mathtt{isBiHom}_X(F)$$

where

$$\mathtt{isBiHom}_X(F) := (f : \Pi_{x:X} A(x)) \to \mathop{\Huge *}_{x:X} (f(x) = \star_{A(x)}) \to F(f) = \star_B$$

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- **Notation:** Given $f : \Pi_{x:X} K_{n(x)}$, let us use $\smile_{x:X} f(x)$ to denote the application of the underlying function of $\smile_{x:X}$ to f.

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Theorem 6 (L., Mörtberg [4])

The map $e : K_0 \times \cdots \times K_m \to (\mathbb{R}P^\infty \to K_m)$ defined by

$$e(x_0, \ldots, x_n) := \lambda\, y\,.\, \sum_{i=0}^{n} t(y)^i \smile x_{n-i}$$

is an equivalence

- Via this equivalence, we get a *total square* $\widehat{\mathsf{Sq}}$ via post-composition

$$\begin{tikzcd} K_m \arrow[rr] \arrow[drr, dashed, "\widehat{\mathsf{Sq}}"'] & & \mathbb{R}P^\infty \to K_{2m} \arrow[d, "\wr"] \\ & & K_0 \times \cdots \times K_{2m} \end{tikzcd}$$

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- ...and thus, we can define our Steenrod Squares $\mathrm{Sq}^n : K_m \to K_{m+n}$ for $0 \leq n \leq m$ via

$$K_m \xrightarrow{\sim} K_0 \times \cdots \times K_{2m} \xrightarrow{\mathrm{pr}_{m+n}} K_{m+n}$$

and $\mathrm{Sq}^n(x) = 0$ otherwise.

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Theorem 7

For any $X, Y : \mathbb{R}P^\infty$ and $n : X \times Y \to \mathbb{N}$ and any unordered quadruple $f : \Pi_{x:X} \Pi_{y:Y} K_{n(x,y)}$, we have

$$\underset{x:X}{\smile} \underset{y:Y}{\smile} f(x,y) = \underset{y:Y}{\smile} \underset{x:X}{\smile} f(x,y)$$

- With our construction of Sq^n in terms of unordered joins, the theorem turns out to be a special case of the following trivial-looking thing

Theorem 8

For any $X, Y : \mathbb{R}P^\infty$ *and* $A : X \times Y \to \mathcal{U}$*, we have a function*

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- Over 2000 lines of code...

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- Does it compute? Absolutely not.

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1 A Brunerie style construction

2 A direct construction

- Idea: Σ-axiom + (b) + (c) suggest a recursive definition
- Let's try to define $\mathrm{Sq}^n_i : K_i \to_\star K_{i+n}$ by induction on i.
- $\underline{i = 0}$. Define $\mathrm{Sq}^0_0(x) = x$ and $\mathrm{Sq}^{n>0}_0(x) = 0$.
- $\underline{i > 0}$. Define

$$\mathrm{Sq}^n_i(x) = \begin{cases} 0 & \text{if } n > i \\ x^2 & \text{if } n = i \\ ? & \text{if } n < i \end{cases}$$

- What to do for '?'.
- Need: map $K_i \to_\star K_{i+n}$ for $n < i$.

Theorem 9

When $n-1 < i$, the map $(K_i \to_\star K_{i+n}) \xrightarrow{\Omega} (\Omega K_i \to_\star \Omega K_{i+n})$ is an equivalence.

Theorem 10 (Wärn '23)

When $n-1 = i$, the fibre of $(K_i \to_\star K_{i+n}) \xrightarrow{\Omega} (\Omega K_i \to_\star \Omega K_{i+n})$ over the dashed map below is contractible.

$$\begin{tikzcd}
\Omega K_i \arrow[r, dashed] & \Omega K_{n+i} \\
K_{i-1} \arrow[u, "\wr"] \arrow[r, "\smile"'] & K_{n+i-1} \arrow[u, "\wr"']
\end{tikzcd}$$

- So, enough to construct $\Omega(\mathrm{Sq}^n) : \Omega K_i \to_\star \Omega K_{i+n}$

$$
\begin{tikzcd}
\Omega K_i \arrow[r, dashed, "\Omega(\mathsf{Sq}^n i)"] \arrow[d, "\wr"'] & \Omega K_{i+n} \arrow[d, "\wr"] \\
K_{i-1} \arrow[r, "\mathsf{Sq}^n_{i-1}"'] & K_{i-1+n}
\end{tikzcd}
$$

Done!

$$\begin{tikzcd}
\Omega K_i \arrow[r, dashed, "\Omega(\mathrm{Sq}^n i)"] \arrow[d, "\wr"'] & \Omega K_{i+n} \arrow[d, "\wr"] \\
K_{i-1} \arrow[r, "\mathrm{Sq}^n_{i-1}"'] & K_{i-1+n}
\end{tikzcd}$$

Done!

- **Pros**:
  - Very HoTT construction
  - Σ-axiom, (b), (c) hold by definition
  - (a) follows easily too

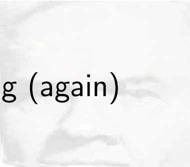
$$\begin{tikzcd} \Omega K_i \arrow[r, dashed, "\Omega(\mathrm{Sq}^n i)"] \arrow[d, "\wr"'] & \Omega K_{i+n} \arrow[d, "\wr"] \\ K_{i-1} \arrow[r, "\mathrm{Sq}^n_{i-1}"'] & K_{i-1+n} \end{tikzcd}$$

Done!

- **Pros**:
  - Very HoTT construction
  - Σ-axiom, (b), (c) hold by definition
  - (a) follows easily too
- **Cons**:
  - Cartan formula – not obvious...
  - Adem relations – ???

Future/Ongoing work

- Finish formalisation of the details...
- Develop the theory of Steenrod algebras in HoTT
- Higher Steenrod powers?
  - Very unclear whether our techniques generalise



Thanks for listening (again)

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