Cellular Homology and the Cellular Approximation Theorem

Axel Ljungström[1], Anders Mörtberg[1], Loïc Pujet[1]

[1]Stockholm University

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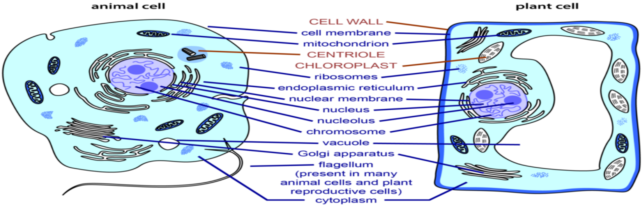

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    - ...proving functoriality via $\underbrace{\text{another homology theory}}_{\text{underdeveloped in HoTT}}$: difficult
- Maybe a more traditional line of attack via *cellular approximation* works?
- This hinges on the theory of CW complexes being developed in HoTT. Interesting in its own right!

Our work

Our primary contributions are

- Proofs of (constructive versions of) the cellular approximation theorem(s) in HoTT
- The construction of a functorial homology theory (à la Buchholtz-Favonia) on the wild category of CW complexes.

Definition 1

A **CW skeleton** $X_\bullet$ is

- an infinite sequence of types and maps $(X_{-1} \xrightarrow{\mathsf{incl}_{-1}} X_0 \xrightarrow{\mathsf{incl}_0} X_1 \xrightarrow{\mathsf{incl}_1} \dots)$ equipped with

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A CW skeleton is said to be **finite (of dimension n)** if incl_i is an equivalence for all $i \geq n$.

- Notation: Let us write $\mathrm{incl}_\infty : X_n \to X_\infty$ for the inclusion into the sequential colimit of $X_\bullet$
- Obvious question: if these were to be the objects of a (wild) category, how should we define arrows?

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- a family $f_i : X_i \to Y_i$ for $i \geq -1$
- a family of homotopies h_i witnessing the commutativity of the following square.

$$\begin{array}{ccc} X_{i+1} & \xrightarrow{f_{i+1}} & Y_{i+1} \\ \uparrow & \overset{h_i}{=\!=} & \uparrow \\ X_i & \xrightarrow[f_i]{} & Y_i \end{array}$$

- The natural notion of homotopies of cellular maps is the following:

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A **cellular homotopy** between cellular maps $f_\bullet, g_\bullet : X_\bullet \to Y_\bullet$, denoted $f_\bullet \sim g_\bullet$, is a family of homotopies h_i witnessing the commutativity of

$$\begin{array}{ccc} Y_i & \xrightarrow{\mathsf{incl}_i} & Y_{i+1} \\ {\scriptstyle g_i}\uparrow & \overset{h_i}{\Longrightarrow} & \uparrow{\scriptstyle \mathsf{incl}_i} \\ X_i & \xrightarrow[f_i]{} & Y_i \end{array}$$

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with fillers, for each $x : X_i$, of the following square of paths.

$$\begin{array}{ccc} \mathsf{incl}_{i+1}(f_{i+1}(\mathsf{incl}_i(x))) & \xrightarrow{h_{i+1}(\mathsf{incl}_i(x))} & \mathsf{incl}_{i+1}(f_{i+1}(\mathsf{incl}_i(x))) \\ \uparrow & & \uparrow \\ \mathsf{incl}_{i+1}(\mathsf{incl}_i(f_i(x))) & \xrightarrow[\mathsf{ap}_{\mathsf{incl}}(h_i(x))]{} & \mathsf{incl}_{i+1}(\mathsf{incl}_i(g_i(x))) \end{array}$$

- Cellular maps is not the only notion of map possible: could also define $\mathrm{Hom}(X_\bullet, Y_\bullet) := (X_\infty \to Y_\infty)$
- This is, in particular, the appropriate definition of map for the category of CW complexes:

Definition 4 (CW complexes)

A type A is a (finite) CW complex if there merely exists a (finite) CW skeleton $X_\bullet$ s.t. $X_\infty \simeq A$.

The (wild) categories at play:

- $\mathtt{CW}^{\mathtt{skel}}$, the category of CW skeleta with cellular maps.
- $\mathtt{CW}$, the category of CW complexes with plain functions as hom-types.
- $\mathtt{CW}^{\mathtt{skel}}_{\infty}$, the category of CW skeleta with $\mathrm{Hom}(X_\bullet, Y_\bullet) := (X_\infty \to Y_\infty)$

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- **Goal: define** $H_n : \mathtt{CW} \to \mathtt{AbGrp}$

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- Define $\partial_{n+1} : \mathtt{deg}(\widehat{\alpha})$
- With some care, get $\partial_0 : \mathbb{Z}[c_1] \to \mathbb{Z}[c_0]$ in a similar fashion

- We get a chain complex...

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- ... and so can define $H_n^{\texttt{skel}} : \texttt{CW}^{\texttt{skel}} \to \texttt{AbGrp}$ by

$$H_n^{\texttt{skel}}(X_\bullet) := \ker \partial_n / \operatorname{im} \partial_{n+1}$$

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Proof.

Standard proof/construction: cellular maps induce chain maps which, in turn, induce maps on homology. □

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- The cellular approximation theorem roughly says that such a section exists.

- In order to stay constructive, we need to restrict ourselves to finite CW skeleta
  - Since $H_n^{\mathsf{skel}_\infty}(X_\bullet) \cong H_n^{\mathsf{skel}_\infty}(X_\bullet^{(n+1)})$ holds trivially, this is not a problem w.r.t. homology.

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*Let $X_\bullet, Y_\bullet$ be CW skeleta **with $X_\bullet$ finite**. Given a map $f : X_\infty \to Y_\infty$, there merely exists a cellular map $f_\bullet : X_\bullet \to Y_\bullet$ s.t. $f_\infty = f$.*

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- Essentially: any map can be cellularly approximated up to dimension n, for any $n \geq 0$.
  - The classical version doesn't require n to be fixed.
  - Maybe a similar statement is still provable in HoTT... (future work)

‘Proof’ of the cellular approximation theorem.

Key components

1. Finite choice: (for commuting Π with truncations)
2. Strengthening ind. hyp. by a further coherence condition
3. $(n-1)$-connectivity of $X_n \to X_\infty$ and $X_n \to X_{n+1}$.

This allows for an explicit inductive construction of $f_\bullet$. □

- Given a map $f : X_\infty \to Y_\infty$ and an $(n+1)$-approximation $\text{approx}_{n+1}(f)$, we can define

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- Problem: the claim that of functoriality of $H_n^{\mathsf{skel}_\infty}$ is a set:
  - the theorem only gives us the mere existence of such $(n+1)$-approximations.
- We need to use the principle of prop-to-set elimination (Kraus [4])

- In our case, this amounts to checking that for any two cellular maps $f_\bullet, g_\bullet : X_\bullet \to Y_\bullet$ s.t. $f_\infty = g_\infty$, we have that $H_n^{\mathtt{skel}}(f_\bullet) = H_n^{\mathtt{skel}}(g_\bullet)$.

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In light of Lemma 8, what we need is a kind of cellular approximation theorem for cellular homotopies.

Theorem 9 (Cellular approximation theorem, part 2)

Given two cellular maps $f_\bullet, g_\bullet : X_\bullet \to Y_\bullet$ with $f_\infty = g_\infty$ and $X_\bullet$ finite, there merely exists a cellular homotopy $f_\bullet \sim g_\bullet$

Proof.

Morally the same as the proof of the first approximation theorem. □

- With this theorem, we have all we need:

Corollary 10

For any two $(n+1)$-approximations $f_\bullet, g_\bullet$ of a map $f_\infty : X_\infty \to Y_\infty$, we have that $H_n^{\mathsf{skel}}(f_\bullet) = H_n^{\mathsf{skel}}(g_\bullet)$

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- The following is easy to see

Proposition 11

$H_n^{\mathsf{skel}_\infty} : \mathtt{CW}_\infty^{\mathtt{skel}} \to \mathtt{AbGrp}$ is a functor

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So far, we have two homology functors:

- $H_n^{\mathsf{skel}} : \mathtt{CW}^{\mathtt{skel}} \to \mathtt{AbGrp}$
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Finally, we would like to extend $H_n^{\mathsf{skel}_\infty}$ this to a functor H_n over $\mathtt{CW}$, the category of spaces with mere CW structures:

$$\mathtt{CW} := \sum_{A:\mathsf{Type}} \| \, \mathtt{CWstr}(A) \, \|_{-1} \qquad \text{where}$$

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- Problem: would like to define $H_n(A) : \mathtt{AbGrp}$ by induction on its mere CW structure $p : \| \mathtt{CWstr}(A) \|_{-1}$ but the universe $\mathtt{AbGrp}$ is a groupoid

- Prop-to-groupoid elimination (Kraus [4]) + SIP + functoriality of $H_n^{\mathsf{skel}_\infty}$

$\implies$ may assume $p : \| \mathtt{CWstr} \|_{-1}$ is on the form $p := | A_\bullet, e |$

- Prop-to-groupoid elimination (Kraus [4]) + SIP + functoriality of $H_n^{\mathsf{skel}_\infty}$

$\Longrightarrow$ may assume $p : \| \texttt{CWstr} \|_{-1}$ is on the form $p := | A_\bullet, e |$

- In this case, we simply define

$$H_n(A) := H_n^{\mathsf{skel}_\infty}(A_\bullet)$$

and we're done

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- And so, finally, we have constructed a functorial **homology theory** $H_n : \texttt{CW} \to \texttt{AbGrp}$
- **...actually, this ongoing work.** We have not yet verified the Eilenberg-Steenrod axioms.

Future/Ongoing work

- Regarding CW complexes:

- Regarding cellular homology:

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  - Approximation of n-connected CW complexes by skeleta with trivial up to dim. $n + 1$

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- Regarding CW complexes:
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- Regarding cellular homology:
    - Prove the Eilenberg-Steenrod axioms
    - Prove the Hurewicz theorem
    - Show that the theory is equivalent to that developed by Graham [3] and Christenssen & Scoccola [2].

Thanks

Thanks for listening!

Appendix: Prop-to-groupoid elimination

Lemma 12

Let $P : A \to \texttt{AbGrp}$ *be a family satisfying*

- *for any* $a, a' : A$*, we have an equivalence* $e_{a,a'} : P(a) \simeq P(a')$
- *for any* a, a', a''*, we have that* $e_{a',a''} \circ e_{a,a'} = e_{a,a''}$

In this case, there is a family $P' : \| A \|_{-1} \to \texttt{AbGrp}$ *s.t.* $P'(| a |) = P(a)$

Proof.

Solution: prop-to-groupoid elimination rule (Kraus [4]) and the structure identity principle. □

[1] Ulrik Buchholtz and Kuen-Bang Hou Favonia. "Cellular Cohomology in Homotopy Type Theory". In: *Proceedings of the 33rd Annual ACM/IEEE Symposium on Logic in Computer Science*. LICS '18. Oxford, United Kingdom: Association for Computing Machinery, 2018, pp. 521–529. ISBN: 9781450355834. DOI: 10.1145/3209108.3209188.

[2] J. Daniel Christensen and Luis Scoccola. "The Hurewicz theorem in Homotopy Type Theory". In: *Algebraic & Geometric Topology* 23 (5 2023), pp. 2107–2140.

[3] Robert Graham. *Synthetic Homology in Homotopy Type Theory*. Preprint. 2018. arXiv: 1706.01540 [math.LO].

[4] Nicolai Kraus. "The General Universal Property of the Propositional Truncation". In: *20th International Conference on Types for Proofs and Programs (TYPES 2014)*. Ed. by Hugo Herbelin, Pierre Letouzey, and Matthieu Sozeau. Vol. 39. Leibniz International Proceedings in Informatics (LIPIcs). Dagstuhl, Germany: Schloss Dagstuhl –

Leibniz-Zentrum für Informatik, 2015, pp. 111–145. ISBN: 978-3-939897-88-0. DOI: 10.4230/LIPIcs.TYPES.2014.111.