Projective Space and Line Bundles in Synthetic Algebraic Geometry

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In this talk:

- SAG at a glance
- projective space $\mathbb{P}^n$
- line bundles, $\mathrm{Pic}(X)$
- classification of line bundles on $\mathbb{P}^n$
- application to $\mathrm{Aut}(\mathbb{P}^n)$

All results are well-known in (external) algebraic geometry, but we present new, synthetic proofs using higher types.

Synthetic algebraic geometry

$$k\text{-}\mathrm{Sch}_{\mathrm{f.p.}} \quad \hookrightarrow \quad \mathrm{Zar}_k := \mathrm{Sh}(k\text{-}\mathrm{Alg}_{\mathrm{f.p.}}{}^{\mathrm{op}}, J_{\mathrm{Zar}})$$

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We interpret HoTT internally in $\mathrm{Zar}_k^{(\infty,1)}$ and write R for the *structure sheaf*:

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- R is a ring.
- Every $x : R$ with $x \neq 0$ is invertible.
  - But we don't have $x = 0 \vee x \neq 0$.
- Every $x : R^n$ with $x \neq 0$ generates a sub-module $\langle x \rangle \subseteq R^n$ with $\langle x \rangle \cong R^1$.
- Every function $f : R \to R$ is a polynomial.
  - But we can't determine $\deg(f) : \mathbb{N}$.
- Every function $R^m \to R^n$ is given by n polynomials in m variables.

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\text{Grassmannian} \quad & \mathbb{G}_k(R^n) := \sum_{P \subseteq R^n \text{ sub-module}} \| P \cong R^k \|
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Interpolating between two points in $\mathbb{P}^n$

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So we have

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We say: $\mathbb{G}_1(\langle p, p' \rangle)$ is the "line" interpolating between p and p'.

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Fix $p_0 \neq p_1$. Then:

$$\begin{array}{rcl} p \neq p_0 & \vee & p \neq p_1 \\ f(p) = f(p_0) & \vee & f(p) = f(p_1) = f(p_0) \end{array}$$

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$$M \otimes M^\vee \to R^1$$

$$R^1 \otimes R^{1\vee} \xrightarrow{\sim} R^1$$

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The *Picard group* of X is

$$\mathrm{Pic}(X) :\equiv \|X \to BR^\times\|_{\mathsf{set}}.$$

Line bundles on $\mathbb{P}^n$

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Q: Are there other line bundles on $\mathbb{P}^n$?

$\mathrm{Pic}(\mathbb{P}^n) = \mathbb{Z}$

Theorem

For every line bundle $L : \mathbb{P}^n \to BR^\times$ *there is a number* $d : \mathbb{Z}$ *such that* $\| L = \mathcal{O}(d) \|$. *Thus:*

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Plan for $n \geq 2$:

- Strengthen the $n = 1$ case to a non-truncated statement.
- Adjust L so that we can expect $\|L = \mathcal{O}(0)\|$.
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Corollary: If $\deg(L) = 0$ then we have $\prod_{p,p' : \mathbb{P}^1} L(p) = L(p')$.

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Thus: $L(p) = L(p')$ for all p, p' on $\mathbb{G}_1(P)$.

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Fix standard points $p_0, p_1 : \mathbb{P}^n$.

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So we conclude: $L = \mathcal{O}(0)$.

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Theorem

Every function $f : \mathbb{P}^m \to \mathbb{P}^n$ is given by $n+1$ homogeneous polynomials of some degree d in $m+1$ variables.

Application: $Aut(\mathbb{P}^n)$

Recall: Every function $R^m \to R^n$ is given by n polynomials in m variables.

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Corollary

Every automorphism $\mathbb{P}^n \xrightarrow{\sim} \mathbb{P}^n$ is given by an invertible matrix, unique up to scalar multiplication.

$$\mathrm{Aut}(\mathbb{P}^n) \cong \mathrm{PGL}_{n+1}(R)$$