Ordinal exponentiation in homotopy type theory

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Ordinals in homotopy type theory

- In the HoTT book, an ordinal is defined as a type X with a prop-valued binary relation $\prec$ that is transitive, extensional and wellfounded.

- Extensionality means that we have

  $$x = y \iff \forall(u : X).(u \prec x \iff u \prec y).$$

  It follows that X is an hset.

- Wellfoundedness is defined in terms of accessibility, but is equivalent to the assertion that for every $P : X \to \mathcal{U}$, we have $\Pi(x : X).P(x)$ as soon as $\Pi(x : X).(\Pi(y : X).(y \prec x \to P(y))) \to P(x)$.

Many other more specialised (and well behaved) notions of ordinals [Martin-Löf 1970; Taylor 1996; Coquand, Lombardi and Neuwirth 2023, ...] , but here we focus on the most general notion.

The ordinal of ordinals

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Similarly, we define $\alpha \leq \beta$ if "α embeds into β without gaps":

$$\alpha \leq \beta :\equiv \Sigma(f : \alpha \xrightarrow{o.p.} \beta).(y \prec f\, x \to \Sigma(x_0 : \alpha).(x_0 \prec x) \times (y = f\, x_0)).$$

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Ord is closed under suprema of arbitrary (small) families of ordinals $\mathsf{sup} : (I \to \mathsf{Ord}) \to \mathsf{Ord}$.

Ordinal arithmetic

$$\alpha + 0 = \alpha$$
$$\alpha + (\beta + 1) = (\alpha + \beta) + 1$$
$$\alpha + \sup \gamma_i = \sup(\alpha + \gamma_i) \qquad \text{(if index set } I \text{ inhabited)}$$

$$\alpha \times 0 = 0$$
$$\alpha \times (\beta + 1) = (\alpha \times \beta) + \alpha$$
$$\alpha \times \sup \gamma_i = \sup(\alpha \times \gamma_i)$$

$$\alpha^0 = 1$$
$$\alpha^{\beta+1} = \alpha^\beta \times \alpha$$
$$\alpha^{\sup \gamma_i} = \sup(\alpha^{\gamma_i}) \qquad \text{(if } I \text{ inhabited, and } \alpha \neq 0)$$
$$0^\beta = 0 \qquad (\text{if } \beta \neq 0)$$

Not a definition, constructively! But a good specification.

Addition and multiplication

For addition and multiplication, there are well known explicit constructions:

$$\langle \alpha + \beta \rangle :\equiv \langle \alpha \rangle + \langle \beta \rangle$$

with inl $a \prec$ inr b, and

$$\langle \alpha \times \beta \rangle :\equiv \langle \alpha \rangle \times \langle \beta \rangle$$

ordered reverse lexicographically:

$$(a, b) \prec (a', b') :\equiv (b \prec b') + ((b = b') \times (a \prec a')).$$

Theorem. The operations $\alpha + \beta$ and $\alpha \times \beta$ satisfy the specifications for addition and multiplication, respectively.

Left division

We can also prove other properties, for example:

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In contrast, Swan [2018] has shown that already

$$\mathbf{2} \times B = \mathbf{2} \times C \implies B = C$$

is not constructively provable for *sets* B and C.

This gives a roundabout proof that not every set can be wellordered, constructively.

What about exponentiation?

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Sierpiński [1958] constructs, for α with a least element $\bot : \alpha$, the exponential α^β as

$$\Sigma(f : \beta \to \alpha).\,\mathsf{supp}(f) \text{ finite}$$

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The order is defined by

$$f \prec g :\equiv f(b^*) \prec_\alpha g(b^*),$$

where b^* is the largest element x such that $f(x) \neq g(x)$ — such b^* exists since by the finite support assumption.

This is not nice, constructively!

Can we do better?

Assume α has a detachable least element, i.e., $\alpha = 1 + \gamma$.

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We can try to make Sierpiński's construction more intensional.

Definition. For ordinals γ and β, let

$$[1 + \gamma]^{\beta} :\equiv \Sigma(xs : \mathsf{List}(\gamma \times \beta)).\,(\mathsf{map\ snd}\ xs)\ \mathsf{decreasing}.$$

- $[1 + \gamma]^{\beta}$ represents a function $\beta \to (1 + \gamma)$ as a list of output-input pairs; elements not in the list are sent to $\mathsf{inl}\ \star$.
- Being strictly decreasing in the second component ensures that each input has at most one output.
- It also ensures that each "function" has at most one representation.

$[1+\gamma]^\beta$ is an ordinal

We can give $[1+\gamma]^\beta$ an order by inheriting the (ordinary) lexicographic order on $\mathsf{List}(\gamma \times \beta)$.

Theorem. (dJKNFX) $[1+\gamma]^\beta$ is an ordinal if γ and β are ordinals.

Remark. In general, the lexicographic order on $\mathsf{List}(\alpha)$ is not wellfounded, but it is for decreasing lists.

$[1 + \gamma]^\beta$ satisfies the specification

<u>**Theorem**</u>. (dJKNFX) $[1 + \gamma]^\beta$ satisfies the specification for the exponential $(1+\gamma)^\beta$.

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Proof sketch.

- $[1+\gamma]^0 = \mathsf{List}(\gamma \times 0) = 1$
- A snd-decreasing list over $\gamma \times (\beta + 1)$ either starts with an element $(\gamma, \mathsf{inr}\ \star)$, or it is snd-decreasing over $\gamma \times \beta$. Hence

$$[1+\gamma]^{\beta+1} = [1+\gamma]^\beta \times (1+\gamma)$$

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- For $[1+\gamma]^{\sup \gamma_i}$, being decreasing in the second component is crucial.

Ordinal exponentiation, in general?

Theorem. $\lambda\beta.(\beta = 0)$ satisfies the specification for the exponential 0^β.

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and P or $\neg P$ holds depending on if $\star : 1$ hits $\mathsf{inl}\ p$ or $\mathsf{inr}\ \star$ for $f : 1 \to P + 1$.

Summary

Ordinals are closed under well behaved addition and multiplication.

New: However, a fully general exponentiation operation is possible if and only if Excluded Middle holds.

The best we can do is $(1+\gamma)^\beta$ and 0^β separately.

Full Agda formalisation.
Building on Escardó's TypeTopology.
https://github.com/fredrikNordvallForsberg/TypeTopology/blob/exponentiation/source/Ordinals/Exponentiation.lagda

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