Two-sided Fibration Categories and Directed Path Objects

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Workshop on Homotopy Type Theory / Univalent Foundations

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Overview

1. Approaches to directed type theory
2. Axiomatising directed fibrations
3. Consequences

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- Syntactic Approaches: Licata 2011, Nuyts 2015, North 2019

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- Probing with an interval: Riehl and Shulman 2017, Weinberger 2022, Weaver and Licata 2020

Semantic Approaches

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  - We want something like Directed Book HoTT: focused on notions of directed path lifting shared by (1-categorical presentations of) many different higher categories
  - We want a theory that works for (∞, n)-categories for all n.

Types are Fibrations

Key insight: Martin-Löf type theory can be modeled by categories with a pullback-stable class of maps called *fibrations*.

Type Theory	Category with fibrations
Types	Fibrations
Terms	Sections
Identity types	Path objects
Path induction	Path lifting

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Is there directed analogue of this table?

Clans

Definition

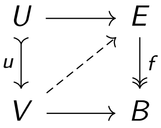
A *clan* is a category $\mathcal{C}$ with a class of morphisms $\mathcal{F}$ called fibrations such that:

- Fibrations are stable under pullback
- $\mathcal{C}$ has a terminal object, and all objects are fibrant

Tribes

Definition

A morphism u in $\mathcal{C}$ is called *anodyne* if it has the left-lifting property w. r. t. fibrations.



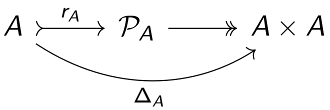
Definition

A *tribe* is a clan with anodyne-fibration factorisations, whose anodyne morphisms are pullback-stable.

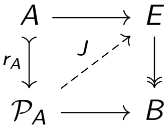
Path Object

Definition

A *path object* for an object A in a tribe is a factorisation of the diagonal $\Delta : A \to A \times A$



The lifting properties of anodyne morphism imply that each commutative diagram as below has a filler.



Examples of tribes

- The category of contexts of a type theory with identity types. Fibrations are dependent projections.
- The category of fibrant objects of a right proper model category whose monomorphisms are cofibrations.

Diclans

Suppose that $\mathcal{C}$ is a tribe, and for each $A \in \mathcal{C}$, let $\mathrm{Fib}_A(\mathcal{C}) \leq C/A$ be the category of fibrations over A in $\mathcal{C}$.

Diclans

We define a *diclan structure on $\mathcal{C}$* so that:

- For each $A \in \mathcal{C}$, and each $\mathfrak{v} \in \{+, -\}$, we have a subcategory

  $$\mathsf{Fib}^{\mathfrak{v}}_A(\mathcal{C}) \hookrightarrow \mathsf{Fib}_A(\mathcal{C})$$

  We refer to the objects in $\mathsf{Fib}^{\mathfrak{v}}_A(\mathcal{C})$ as $\mathfrak{v}$-fibrations, and morphisms as $\mathfrak{v}$-cartesian morphisms.

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- These inclusions create terminal objects, pullbacks of fibrations, and path objects.

Diclans

- The $\mathfrak{v}$-fibrations and their morphisms are pullback-stable. For each $f : A \to B$, we have a functor

$$\mathrm{Fib}^{\mathfrak{v}}_A(\mathcal{C}) \xleftarrow{\quad \Sigma_f \quad} \mathrm{Fib}^{\mathfrak{v}}_B(\mathcal{C})$$

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- N.B. These functors need not be adjoint.
- We often ask that $\mathrm{Fib}^{\mathfrak{d}}_A(\mathcal{C})$ be a replete subcategory of $\mathrm{Fib}_A(\mathcal{C})$.

Diclans

- Basic results about diclans have analogues in clans. e.g. diclans have products

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- For each variance $\mathfrak{v}$, we obtain a comprehension category structure

$$\begin{array}{ccc} \mathrm{Fib}^{\mathfrak{v}}(\mathcal{C}) & \hookrightarrow & \mathcal{C}^{\to} \\ & \searrow \qquad \swarrow_{s} & \\ & \mathcal{C} & \end{array}$$

Iterating fibrations

- If $\mathcal{C}$ is a clan, then $\mathrm{Fib}_A(\mathcal{C})$ is a clan. Fibrations in $\mathrm{Fib}_A(\mathcal{C})$ are fibrations in $\mathcal{C}$.

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- We can take $+$-fibrations in $\mathrm{Fib}^+_A(\mathcal{C})$ to be $+$-fibrations in $\mathcal{C}$.
- What are $-$-fibrations in $\mathrm{Fib}^+_A(\mathcal{C})$?

Iterating fibrations

We require a coinductive structure on a diclan $\mathcal{C}$ so that:

- For each object $A \in \mathcal{C}$, we have a diclan structure on $\mathsf{Fib}_A(\mathcal{C})$ and $\mathsf{Fib}_A^{\mathfrak{v}}(\mathcal{C})$ for $\mathfrak{v} \in \{+, -\}$.

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- The inclusions $\mathrm{Fib}_A^{\mathfrak{v}}(\mathcal{C}) \hookrightarrow \mathrm{Fib}_A(\mathcal{C})$ preserve diclan structure.
- Base change functors preserve diclan structure.

Compatibility Conditions

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- For each $\mathfrak{v}$-fibration $f : E \twoheadrightarrow A$, we have

$$\mathrm{Fib}^+_f(\mathrm{Fib}^+_A(\mathcal{C})) = \mathrm{Fib}^+_E(\mathcal{C}).$$

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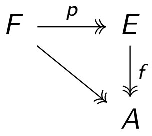
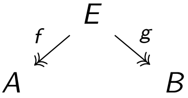
$$\mathrm{Fib}^{+}_{f}(\mathrm{Fib}^{+}_{A}(\mathcal{C})) = \mathrm{Fib}^{+}_{E}(\mathcal{C}).$$

  Thus, the arrow p in the diagram below is a $\mathfrak{v}$-fibration over E if and only if it is a $\mathfrak{v}$-fibration over f in $\mathrm{Fib}^{\mathfrak{v}}_{A}(\mathcal{C})$.



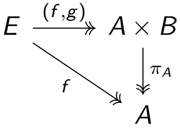
Compatibility Conditions

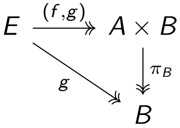
- Given a span of fibrations



  the following are equivalent:

  - f is a $-$-fibration over A, and (f, g) is a $+$-fibration over the projection π_A in $\mathrm{Fib}^-_A(C)$.
  - g is a $+$-fibration over B, and (f, g) is a $-$-fibration over the projection π_B in $\mathrm{Fib}^+_A(C)$.





Compatibility Conditions

- We also require a condition on morphisms of two-sided fibrations.

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- We also require a condition on morphisms of two-sided fibrations.
- Thus, we have a notion of *two-sided fibration*.

Comprehension towers

For each sequence of variances $\mathfrak{v}_1, \ldots, \mathfrak{v}_n$, we obtain a tower of Grothendieck fibrations and cartesian functors:

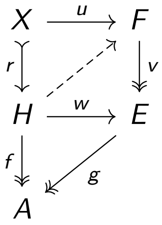
$$
\begin{array}{ccc}
\mathsf{Fib}^{\mathfrak{v}_1,\ldots,\mathfrak{v}_n}(\mathcal{C}) & \hookrightarrow & \mathcal{C}^{\to\cdots\to} \\
\downarrow & & \downarrow \\
\vdots & \hookrightarrow & \vdots \\
\downarrow & & \downarrow \\
\mathsf{Fib}^{\mathfrak{v}_1,\mathfrak{v}_2}(\mathcal{C}) & \hookrightarrow & \mathcal{C}^{\to\to} \\
\downarrow & & \downarrow \\
\mathsf{Fib}^{\mathfrak{v}_1}(\mathcal{C}) & \hookrightarrow & \mathcal{C}^{\to} \\
\downarrow & \swarrow_{s} & \\
\mathcal{C} & &
\end{array}
$$

Directed Path Objects

Directed Path Objects

Definition

A $\mathfrak{v}$*-sprout over* A *in* $\mathcal{C}$ is a pair $X \xrightarrow{r} H \overset{f}{\twoheadrightarrow} A$ where f is a $\mathfrak{v}$-fibration. We say that a $\mathfrak{v}$-sprout is a $\mathfrak{v}$*-cofibration* if for each commutative diagram of solid arrows



if f, g, fr, vg are $\mathfrak{v}$-fibrations, v, w are $\mathfrak{v}$-cartesian, and v is a fibration, then we can find a $\mathfrak{v}$-cartesian filler.

Directed Path Objects

Definition

A *dipath object on A* is a factorisation of the diagonal

$$A \overset{r_A}{\rightarrowtail} \mathcal{H}_A \overset{(s_A,t_A)}{\twoheadrightarrow} A \times A$$

$$\Delta_A$$

such that (s_A, t_A) is a two-sided fibration, (r_A, s_A) is a $-$-cofibration, and (r_A, t_A) is a $+$-cofibration.

Ditribes

Definition

A *ditribe* is a diclan $\mathcal{C}$ such that:

- Each object A has a dipath object.
- For each $x : X \to A$, let $r_{x,-}, s_{x,-}, t_{x,-}$ be defined using the pullback

$$\begin{array}{ccc}
X & \xrightarrow{\quad x \quad} & A \\
{\scriptstyle r_{x,-}} \downarrow & \lrcorner & \downarrow {\scriptstyle r_A} \\
\mathcal{H}_A(x,-) & \longrightarrow & \mathcal{H}_A \\
{\scriptstyle s_{x,-} \times t_{x,-}} \downarrow & \lrcorner & \downarrow {\scriptstyle (s_A, t_A)} \\
X \times A & \xrightarrow[x \times A]{} & A \times A
\end{array}$$

Then, $(r_{x,-}, t_{x,-})$ is a $+$-cofibration.

Ditribes

- For each $x : X \to A$, let $r_{-,x}, s_{-,x}, t_{-,x}$ be defined using the pullback

$$
\begin{array}{ccc}
X & \xrightarrow{\quad x \quad} & A \\
{\scriptstyle r_{-,x}} \downarrow & \lrcorner & \downarrow {\scriptstyle r_A} \\
\mathcal{H}_A(x, -) & \longrightarrow & \mathcal{H}_A \\
{\scriptstyle s_{-,x} \times t_{x,-}} \downarrow & \lrcorner & \downarrow {\scriptstyle (s_A, t_A)} \\
A \times X & \xrightarrow[A \times x]{} & A \times A
\end{array}
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  Then, $(r_{x,-}, s_{x,-})$ is a $-$-cofibration.

In the intuitive type theory, this is based path induction. If we fix the source, we can eliminate into covariant type families. If we fix the target, we can eliminate into contravariant type families.

Ditribes

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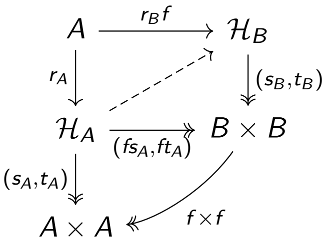
- Base change functors preserves ditribe structure.
- Identity arrows are pseudo-natural: for each $f : A \to B$, the $+$-cartesian and $-$-cartesian solutions to the following lifting problem agree:



Examples

- The category of categories. $+$- and $-$-Fibrations are Grothendieck (op)fibrations. Dipath objects are arrow categories.
- More generally, any 2-category with finite limits.

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- More generally, any 2-category with finite limits.
- The category of quasicategories. $+$- and $-$-fibrations are (co)cartesian fibrations. Dipath objects are arrow $(\infty, 1)$-categories.
- More generally, any ∞-cosmos induced by a model category whose monomorphisms are cofibrations.

Examples (WIP)

- The category of 2-categories. $+$- and $-$-fibrations are normal 2-fibrations. Dipath objects are lax arrow categories.

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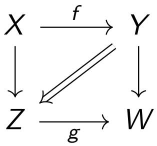

- We can probably add some adjectives. e.g. "split".
- The category of strict n-categories. $+$- and $-$-fibrations are normal n-fibrations.

Examples (WIP)

- We can "reverse the polarity". For each ditribe $\mathcal{C}$, there is a ditribe $\mathcal{C}^{\mathsf{co}}$ whose $+$-fibrations are $-$-fibrations in $\mathcal{C}$ and vice versa.

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- We can “reverse the polarity”. For each ditribe $\mathcal{C}$, there is a ditribe $\mathcal{C}^{\mathrm{co}}$ whose $+$-fibrations are $-$-fibrations in $\mathcal{C}$ and vice versa.
- In most cases, the fibrations are generated by the dipath objects.

First consequences

- Directed path objects are essentially unique.

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- Directed path objects of compound types can be described using their parts. e. g. directed paths of pairs are pairs of direct paths.

Types are weak ω-categories

- The lifting properties induce a notion of directed path composition.

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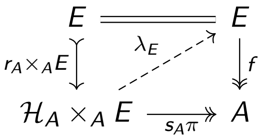
- The lifting properties induce a notion of directed path composition.
- The tower of directed path objects $A, \mathcal{H}_A, \mathcal{H}_{\mathcal{H}_A}, \ldots$ has the structure of a globular ω-category.

Directed fibrations are higher Grothendieck fibrations

The $+$- and $-$-fibrations satisfy lifting properties analogous to those of Grothendieck fibrations.

- If $f : E \twoheadrightarrow A$ is a $-$-fibration, then we have a $-$-cartesian filler:



- Intuitively, if we have $\phi : a' \to a$ and $e : E$ such that $f(e) = a$, then $f\lambda_E(\phi, e) = a'$.

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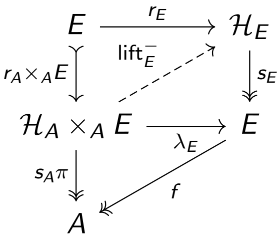
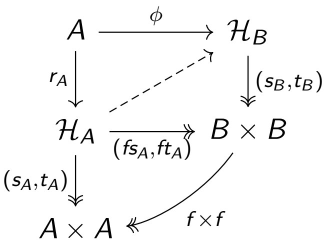


- Intuitively, if we have $\phi : a' \to a$ and $e : E$ such that $f(e) = a$, then $s_E \,\text{lift}^-_E(\phi, e) = \lambda_E(\phi, e)$. A calculation in $\text{Fib}_A(\mathcal{C})$ implies that

$$\text{lift}^-_E(\phi, e) : \lambda_E(\phi, e) \to e$$

Two-sided lifting conditions

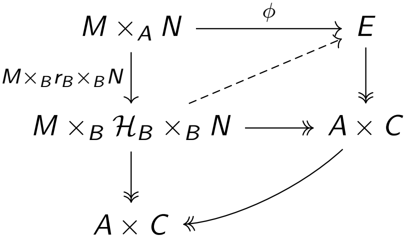
- Other path lifting principles follow from the axioms.
- Suppose that $E \twoheadrightarrow B \times C$ is a two-sided fibration. We can characterize those ϕ such that the following sort of lifting problem admit solutions that are both $+$- and $-$-cartesian.



- In general ϕ behaves like a lax natural transformation. For the problem above, we need ϕ which behave like pseudo-natural transformations.

Two-sided lifting conditions

- Suppose that $M \twoheadrightarrow A \times B$, $N \twoheadrightarrow B \times C$, $E \twoheadrightarrow A \times C$ are two-sided fibrations.



We can characterize those ϕ such that lifting problems of this form have a solution that is both $+$-cartesian and $-$-cartesian.

- Such ϕ are *equivariant*: they respect both the covariant and the contravariant action of $\mathcal{H}_B$.

Thank you.