Tangent bundles and Euler classes

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Overview and preliminaries

Overview:

- Euler classes of oriented sphere bundles
- Whitney sum formula – ⚠ – WIP
- Comparison with Thom class
- Tangent sphere bundles of spheres
- Outlook

For any type T, we have the classifying types of T-bundles,

$$\mathrm{BAut}(T) :\equiv \sum_{X:\mathcal{U}} \|X \simeq T\|,$$

and of *oriented* T-bundles,

$$\mathrm{BAut}_1(T) :\equiv \sum_{X:\mathcal{U}} \|X \simeq T\|_0.$$

A pointed type T is *central*, if evaluation at the base point induces an equivalence:

$$\mathrm{ev}_{\mathrm{pt}_T} : (T \to T)_{(\mathrm{id}_T)} \xrightarrow{\sim} T.$$

If T is central, then for any $(X, \omega) : \mathrm{BAut}_1(T)$, evaluation at the base point gives an equivalence

$$\big((T, |\mathrm{id}|_0) = (X, \omega)\big) \xrightarrow{\sim} X.$$

Euler classes

We can *define* (up to resizing):

$$\mathrm{B}^n\mathbb{Z} :\equiv \mathrm{BAut}_1(\mathrm{B}^{n-1}\mathbb{Z})$$
$$\mathrm{B}\mathcal{G}(n) :\equiv \mathrm{BAut}(\mathrm{S}^{n-1})$$
$$\mathrm{B}\mathcal{SG}(n) :\equiv \mathrm{BAut}_1(\mathrm{S}^{n-1}).$$

The *universal Euler class* is the map

$$\mathrm{e}_n^{\mathbb{Z}} : \mathrm{BAut}_1(\mathrm{S}^{n-1}) \to \mathrm{BAut}_1(\mathrm{B}^{n-1}\mathbb{Z})$$

induced by $(n-1)$-truncation for $n > 1$. Using notations above, $\mathrm{e}_n^{\mathbb{Z}} : \mathrm{B}\mathcal{SG}(n) \to \mathrm{B}^n\mathbb{Z}$.
The direct sum of two oriented sphere bundles is given by the join (with suitable orientation):

$$\oplus : \mathrm{B}\mathcal{SG}(n) \to_* \mathrm{B}\mathcal{SG}(m) \to_* \mathrm{B}\mathcal{SG}(n+m)$$
$$\bar{X} \oplus \bar{Y} :\equiv (X \star Y, \_)$$

The *Whitney sum formula* states

$$\mathrm{e}_{n+m}(\bar{X} \oplus \bar{Y}) = \mathrm{e}_n(\bar{X}) \smile \mathrm{e}_m(\bar{Y})$$

This is automatic if we can define the cup product using joins:

$$\smile : \mathrm{B}^n\mathbb{Z} \to_* \mathrm{B}^m\mathbb{Z} \to_* \mathrm{B}^{n+m}\mathbb{Z}$$
$$\bar{X} \oplus \bar{Y} :\equiv (\|X \star Y\|_{n+m-1}, \_)$$

Notice the level shift compared to using truncation as the definition,

$$|\_|_n : \mathrm{S}^n \to_* \mathrm{B}^n\mathbb{Z} :\equiv \|\mathrm{S}^n\|_n$$

with the cup product induced by the smash product

$$\mathrm{S}^n \to_* \mathrm{S}^m \to_* \mathrm{S}^n \wedge \mathrm{S}^m \simeq_* \mathrm{S}^{n+m}$$

Whitney sum formula (sketch)

Another way of defining the cup product is via delooping:
By induction on n, we use the equivalence

$$\begin{aligned} &\mathrm{B}^{n+1}\mathbb{Z} \to_* \mathrm{B}^m\mathbb{Z} \to_* \mathrm{B}^{n+1+m}\mathbb{Z} \\ \simeq\ &\Omega\mathrm{B}^{n+1}\mathbb{Z} \to_* \mathrm{B}^m\mathbb{Z} \to_* \Omega\mathrm{B}^{n+1+m}\mathbb{Z} \\ \simeq\ &\mathrm{B}^n\mathbb{Z} \to_* \mathrm{B}^m\mathbb{Z} \to_* \mathrm{B}^{n+m}\mathbb{Z} \end{aligned}$$

to define $\smile_{n+1,m}$ in terms of $\smile_{n,m}$. For the base case $n \equiv 0$, we use an induction on m.

To compare with the join definition, we need in the step case a pointed homotopy:

$$\begin{array}{ccc} \Omega\mathrm{B}^{n+1}\mathbb{Z} & \xrightarrow{\Omega(\star_{n+1,m})} & \Omega(\mathrm{B}^m\mathbb{Z} \to_* \mathrm{B}^{n+1+m}\mathbb{Z}) \\ \simeq\downarrow & & \downarrow\simeq \\ \mathrm{B}^n\mathbb{Z} & \xrightarrow{\star_{n,m}} & (\mathrm{B}^m\mathbb{Z} \to_* \mathrm{B}^{n+m}\mathbb{Z})\,. \end{array}$$

Since the bottom right type is an H-space, it's enough to give an unpointed homotopy (thanks, Evan!). Fix $\bar{Y} : \mathrm{B}^m\mathbb{Z}$. We need the outer square below to commute:

$$\begin{array}{ccc} \Omega\mathrm{B}^{n+1}\mathbb{Z} & \xrightarrow{\Omega(-\star_{n+1,m}\bar{Y})} & \Omega\,\mathrm{BAut}_1(\|\mathrm{B}^n\mathbb{Z}\star Y\|_{n+m}) \\ \simeq\downarrow & & \downarrow\simeq \\ \mathrm{B}^n\mathbb{Z} & \xrightarrow{|\mathrm{inl}\,-|} & \|\mathrm{B}^n\mathbb{Z}\star Y\|_{n+m} \\ \equiv\downarrow & & (-=\mathrm{pt})\downarrow\simeq \\ \mathrm{B}^n\mathbb{Z} & \xrightarrow{-\star_{n,m}\bar{Y}} & \mathrm{BAut}_1(\|\mathrm{B}^{n-1}\mathbb{Z}\star Y\|_{n+m-1}) \end{array}$$

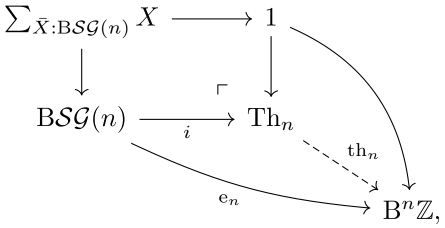
The bottom square boils down to equivalences

$$\|X\star Y\|_{n+m-1} \simeq \left(|\mathrm{inl}\,\bar{X}| =_{\|\mathrm{B}^n\mathbb{Z}\star Y\|_{n+m}} |\mathrm{inl}\,\bar{\mathrm{B}}^{n-1}\mathbb{Z}|\right),$$

with a compatibility when $\bar{X}$ is the base point.

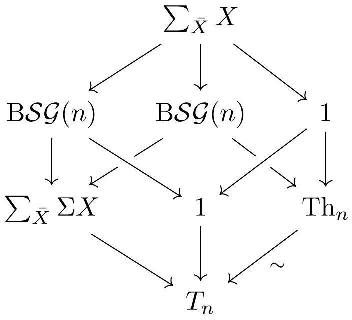
Thom classes

The universal Thom space is the Thom space of the universal oriented n-dimensional bundle,



and we want the universal Thom class th_n making the diagram commute.

We have a map of spans inducing a commuting cube,



where the front left and the back squares are pushouts. Then the front right is as well, and this gives an equivalent definition of the Thom space.

Thom class continued

We get an equivalence

$$(\mathrm{Th}_n \to_* \mathrm{B}^n\mathbb{Z}) \simeq \prod_{\bar{X}} (\Sigma X \to_* \mathrm{B}^n\mathbb{Z}),$$

so we can define the Thom class in the RHS, and since $(\Sigma X \to_* \mathrm{B}^n\mathbb{Z})$ is a set, it suffices to let the map $\Sigma \mathrm{S}^{n-1} \to_* \mathrm{B}^n\mathbb{Z}$ be the loop–suspension adjunct of the composite generator:

$$\mathrm{S}^{n-1} \to_* \mathrm{B}^{n-1}\mathbb{Z} \to_* \Omega\mathrm{B}^n\mathbb{Z}.$$

To check that this restricts to the Euler class when restricted along $i : \mathrm{B}\mathcal{SG}(n) \to \mathrm{Th}_n$, it's useful to note a further equivalence

$$(\Sigma X \to_* \mathrm{B}^n\mathbb{Z}) \simeq \sum_{\bar{Y}:\mathrm{B}^n\mathbb{Z}} \big(X \to (\bar{\mathrm{B}}^{n-1}\mathbb{Z} = \bar{Y})\big),$$

where we have the map $\bar{X} \mapsto (\mathrm{e}_n^{\mathbb{Z}}(\bar{X}), q)$, where q is the composite equivalence

$$X \to \|X\|_n \to \big(\mathrm{pt} = \mathrm{e}_n^{\mathbb{Z}}(\bar{X})\big).$$

These definitions agree: To check this, note that this is a proposition, so it suffices to check the base point.

Tangent sphere bundles of spheres

It's convenient for us to define $\mathrm{S}^n :\equiv \{\pm 1\}^{\star n+1}$, where $\mathrm{S}^0 \simeq \{\pm 1\}$ is an H-space under multiplication, pointed at $+1$. This is equivalent to the suspension definition, since generally $\mathrm{S}^0 \star Y \simeq \Sigma Y$.

A *reflection* on a type A is an equivalence $r : A \simeq A$ together with a homotopy $h : \mathrm{id}_A * r = \mathrm{sym}_{A,A}$.
A *coherent reflection* is one equipped with paths

$$h(\mathrm{inl}\, a) =_{\mathrm{inl}\, a = \mathrm{inr}\, a} \mathrm{glue}(a, a)$$
$$h(\mathrm{inr}\, a) =_{\mathrm{inr}(r(a)) = \mathrm{inl}\, a} \mathrm{glue}(a, r(a))^{-1}$$

for each $a : A$.
The 0-sphere S^0 has a coherent reflection given by swapping the two points.

Fact: S^0 has a coherent reflection (multiplication by -1). Intuition: the matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \sim \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

are homotopic via by counter-clockwise rotation.

We want to define the tangent sphere bundles of spheres $\tau^{n+1} : \mathrm{S}^n \to \mathrm{B}\mathcal{G}(n)$ along with equivalences $\theta^{n+1}(x) : \mathrm{S}^0 \star \tau^{n+1}(x) \simeq \mathrm{S}^n$. To also handle projective spaces, and with a hope to giving attaching maps for the cell structure of Grassmannians, we generalize.

Tangent bundles of joins of torsors

Fix a type A. An A-*torsor* is a type E with a map $t : E \to (A \simeq E)$.
By functoriality of join, this induces scalar multiplication maps $\cdot : A \to E^{\star n} \to E^{\star n}$.

If A has a reflection r, and E is an A-torsor, then for all $n : \mathbb{N}$ and $x : E^{\star n}$, we have a type $\tau^n(x)$ (merely equivalent to $E^{\star n-1}$) and an equivalence $\theta^n(x) : A \star \tau^n(x) \simeq E^{\star n}$.

For $n = 0$, there's nothing to do. For $x : E \star E^{\star n}$, we induct on x, defining:

$$\tau^{n+1}(\mathrm{inl}(e)) :\equiv E^{\star n}$$
$$\tau^{n+1}(\mathrm{inr}(x)) :\equiv E \star \tau^n(x).$$

On $\mathrm{glue}(e, x)$, we take the composition

$$E^{\star n} \xlongequal{\theta^n(x)^{-1}} A \star \tau^n(x) \xlongequal{r^{-1} \star \mathrm{id}} A \star \tau^n(x) \xlongequal{t(e) \star \mathrm{id}} E \star \tau^n(x).$$

For the point constructors, we define:

$$\theta^{n+1}(\mathrm{inl}(e)) :\equiv A \star E^{\star n} \xrightarrow{t(e) \star \mathrm{id}} E^{\star (n+1)}$$
$$\theta^{n+1}(\mathrm{inr}(x)) :\equiv A \star (E \star \tau^n(x)) \xrightarrow{\mathrm{twist}} E \star (A \star \tau^n(x)) \xrightarrow{\mathrm{id} \star \theta^n(x)} E^{\star (n+1)}.$$

The glue case amounts to (use twist):

$$\begin{array}{ccc}
A \star (A \star \tau^n(x)) & \xrightarrow{\mathrm{id} \star (t(e) \star \mathrm{id})} & A \star (E \star \tau^n(x)) \\
\mathrm{id} \star (r \star \mathrm{id}) \downarrow & & \downarrow \mathrm{twist} \\
A \star (A \star \tau^n(x)) & \xrightarrow{t(e) \star \mathrm{id}} & E \star (A \star \tau^n(x)) \\
\mathrm{id} \star \theta^n(x) \downarrow & & \downarrow \mathrm{id} \star \theta^n(x) \\
A \star E^{\star n} & \xrightarrow{t(e) \star \mathrm{id}} & E \star E^{\star n}\,.
\end{array}$$

Hairy ball theorem

If the reflection is coherent, then we have paths

$$\theta^n(e)(\mathrm{inl}\, a) = a \cdot e \quad \text{in } E^{\star n}.$$

We get the *Hairy ball theorem*: If n is even, then the tangent bundle of S^n has no section.

Proof. Assume $s : \prod_x \tau^{n+1}(x)$. Note that S^0 has a neutral element $+1$. For any x we have a path

$$x = 1{\cdot}x = \theta^n(x)(\mathrm{inl}\, 1) = \theta^n(x)(\mathrm{inl} -1) = -1{\cdot}x.$$

Since n is even $-1 \cdot \_$ equals $r \star \mathrm{id}$, so the identity is homotopic to a map of degree -1.

We're also working on a proof via Euler classes: Let $n > 0$ and let $E : B \to \mathrm{B}\mathcal{SG}(n)$ be an oriented sphere bundle on a type B. If E merely has a section, then $\mathrm{e}_n^{\mathbb{Z}}(E) = 0$.

Let $R : \mathrm{S}^n \to \mathcal{U}$ be a family of $(n-1)$-connected types for $n \geq 0$. Then R merely has a section. In particular, for $k \geq n$, any k-sphere bundle on a n-sphere merely has a section.

Thanks!