Strict Rezk completions of models of type theory

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Motivation

Motivation: prove properties of type theories and their models.

Computation up to judgemental equality Canonicity, normalization.

Computation up to homotopy Homotopy canonicity, normalization.

Comparison between type theories Coherence, conservativity, embedding, internal language theorems and conjectures.

Models of type theory: categories with families + additional structure.

Higher models of type theory

Natural to consider **higher** models of type theory.

Components of (strict) models: sets (Ob, Hom, Ty, Tm).
Components of higher models: ∞-groupoids (Ob_∞, Hom_∞, Ty_∞, Tm_∞).

More natural statements of some properties:

Homotopy canonicity The ∞-groupoid $\mathrm{Tm}_\infty(1, \mathrm{Bool})$ is equivalent to $\{\mathrm{true}, \mathrm{false}\}$.

Homotopy normalization Every term has a contractible ∞-groupoid of normal forms.

Coherence Some ∞-groupoids of terms are 0-truncated.

Higher models of type theory

∞-categorical definitions of higher models:

Kraus (2021): ∞-categories with families.

Nguyen and Uemura (2022): ∞-type theories, with ∞-category of models.

Problems:

- Requires ∞-categorical tools.
- Comparison between (strict) models and higher models?
  $\rightsquigarrow$ coherence issues.

Goals of this talk

- Explain how to use cubical models as a notion of higher model.
  Cubical model $\neq$ cubical set model.
  Cubical model $=$ internal model of HoTT in cubical sets.
- Strict Rezk completion:
  Provides a comparison between strict models and univalent higher models.
  In cubical sets: use the same tools as for univalent universes.

What is needed for applications?

Given a model $\mathcal{M}$ of HoTT (or some other type theory):

- Define ∞-groupoids $\mathcal{M}.\mathrm{Ob}_\infty$, $\mathcal{M}.\mathrm{Hom}_\infty$, $\mathcal{M}.\mathrm{Ty}_\infty$, $\mathcal{M}.\mathrm{Tm}_\infty$.
  1-cells in $\mathcal{M}.\mathrm{Tm}_\infty$ are identifications between terms, etc.
- For some good notion of ∞-groupoid (e.g. types in a model of HoTT).
- Extend the type and term formers to ∞-functors.
- Additional structure?
- Keep the definitional equalities of the original model $\mathcal{M}$?
  This is needed to construct new models.

This also makes sense for other algebraic structures, e.g. categories.

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The homotopy theory of type theories

Kapulkin and Lumsdaine (2018) construct a left semi-model structure on (contextual) models of type theory. (With Id, Σ and Π)

Maps in the associated ∞-category of the form

$$\mathbf{0}[\boldsymbol{\Gamma} \vdash \boldsymbol{a} : \boldsymbol{A}] \to \mathcal{M}$$

essentially give the ∞-groupoid of terms of $\mathcal{M}$.

But we don't have enough definitional equalities.

Rezk completion of categories

Take a category $\mathcal{C}$ (in **Set**). We have $\Delta : \mathbf{Set} \to \mathbf{cSet}$.

- $\mathcal{C}' \triangleq \Delta(\mathcal{C})$ is a strict category internally to **cSet**.
  (Components are 0-truncated fibrant cubical sets/families.)
- Its **Rezk completion** $\mathrm{RC}(\mathcal{C}')$ is a **saturated**[1] category.
  $\rightsquigarrow$ Components are fibrant cubical sets/families,
  with the "correct" homotopy type.
- There is a weak equivalence $i : \mathcal{C}' \to \mathrm{RC}(\mathcal{C}')$.
  $\rightsquigarrow$ They satisfy the same "categorical" properties.
  (e.g. properties expressible in FOLDS.)

[1] univalent

Rezk completion of models of type theory ?

Can we do the same for models of type theory?

- Take a model $\mathcal{M}$.
- View it a strict model internally to a model of HoTT (e.g. **cSet**).
- Construct some Rezk completion $\mathrm{RC}(\mathcal{M})$?
- The components of $\mathrm{RC}(\mathcal{M})$ now correspond to the correct ∞-groupoids?

Cannot stay in pure HoTT

Definable in HoTT:

- Strict models of type theory.

Not (known to be) definable in HoTT (same as semi-simplicial types):

- Untruncated models of type theory.
- Saturated models of type theory.

Possible solutions:

1. Use ∞-categories with families in two-level type theory.
2. Use untruncated categories with families with strict CwF laws.

Cartesian cubical sets

In the internal language of **cSet** $=$ **Psh**$(\square)$:

- Universes of sets Set, with strict equality $(=)$.
  Models extensional type theory.
- Universes of fibrant sets $\mathrm{Set}_{\mathsf{fib}}$, with paths $(\sim)$.
  Models HoTT.

Cubical models

Equivalent definitions of "cubical model of type theory":

- Models in the internal language of **cSet**.
- Models valued in cubical sets $\mathbf{Mod}_{\mathsf{HoTT}}(\mathbf{cSet})$.
- Cubical presheaves of set-valued models $[\square^{\mathrm{op}}, \mathbf{Mod}_{\mathsf{HoTT}}]$.

(Holds for any essentially algebraic theory $\mathcal{T}$).

Saturation for cubical models

Definition

A cubical model of type theory has **fibrant components** if $\mathrm{Tm}(\Gamma, A)$ is a fibrant family. (Assuming $\mathrm{Ty}_n(\Gamma) \cong \mathrm{Tm}(\Gamma, \mathcal{U}_n)$)

Definition

A cubical model of type theory is **saturated** if

$$(x : \mathrm{Tm}(\Gamma, A)) \to \text{is-contr}((y : \mathrm{Tm}(\Gamma, A)) \times (\mathrm{Tm}(\Gamma, \mathrm{Id}_A(x, y))).$$

Equivalently:

$$\text{is-equiv}((x \sim y) \xrightarrow{\text{path-to-id}} \mathrm{Tm}(\Gamma, \mathrm{Id}_A(x, y))).$$

Strict Rezk completion of a category

Let $\mathcal{C}$ be a cubical category.

Definition

A strict Rezk completion of $\mathcal{C}$ is a functor $i : \mathcal{C} \to \overline{\mathcal{C}}$ such that:

- The cubical category $\overline{\mathcal{C}}$ has **fibrant components**.
- The cubical category $\overline{\mathcal{C}}$ is **saturated**.

$$(x \in \overline{\mathcal{C}}) \to \text{is-contr}((y \in \overline{\mathcal{C}}) \times (x \cong y)).$$

- The external functor $1^*_\square(i)$ is a **(split) weak equivalence**.

Strict Rezk completion of a cubical model of type theory

Let $\mathcal{M}$ be a cubical model.

Definition

A strict Rezk completion of $\mathcal{M}$ is a morphism $i : \mathcal{M} \to \overline{\mathcal{M}}$ such that:

- The cubical model $\overline{\mathcal{M}}$ has **fibrant components**.
- The cubical model $\overline{\mathcal{M}}$ is **saturated**.

$$(x : \mathsf{Tm}(\Gamma, A)) \to \text{is-contr}((y : \mathsf{Tm}(\Gamma, A)) \times (\mathsf{Tm}(\Gamma, \mathsf{Id}_A(x, y))).$$

- The external functor $1^*_\square(i)$ is a **(split) weak equivalence**.

Construction of strict Rezk completion in cubical sets

Main ideas:

- Models of HoTT are algebras of a complicated generalized algebraic theories. $\rightsquigarrow$ Generalize from the case of simpler generalized algebraic theories.
  (e.g. propositions, preorders, E-categories, etc.).
- Reuse the tools of the construction of cubical set models,
  In particular the construction of univalent universes using Glue-types.
  (i.e. using the equivalence extension structure)

Axioms for cubical set models

Following (Orton and Pitts 2018), (Licata, Orton, Pitts, and Spitters 2018), (Angiuli, Brunerie, Coquand, Harper, (Favonia), and Licata 2021), (Cavallo, Mörtberg, and Swan 2020).

Interval $\mathbb{I} : \mathrm{Set}$, with endpoints $0, 1 : \mathbb{I}$.

$\rightsquigarrow$ Maps $(\mathbb{I}^n \to X)$ are n-dimensional cubes in X.

Cofibration classifier $\mathrm{Cof} \hookrightarrow \Omega$, closed under $\top$, $\bot$, $\wedge$, $\vee$, $\forall_{\mathbb{I}}$ and $(- =_{\mathbb{I}} -)$.

$\rightsquigarrow$ Maps $((i : \mathbb{I}^n) \to [\alpha(i)] \to X)$ are partial n-dimensional cubes in X.

Extension structures

A set X has an extension structure if every partial element can be extended to a total element.

$$\mathsf{HasExt}(X) \triangleq \forall(\alpha : \mathsf{Cof})(x : [\alpha] \to X) \to \{X \mid \alpha \hookrightarrow x\}.$$

Sets with extension structures correspond to **trivially fibrant** sets.

A set is trivially fibrant iff it is fibrant and contractible.

Universal property of the Rezk completion

Universal property of $i : \mathcal{C} \to \mathrm{RC}(\mathcal{C})$:

Every functor $F : \mathcal{C} \to \mathcal{D}$ into a saturated category $\mathcal{D}$ factors uniquely through $i : \mathcal{C} \to \mathrm{RC}(\mathcal{C})$.

$\rightsquigarrow$ Define the strict Rezk completion $\overline{\mathcal{C}}$ as a quotient higher inductive type?

$$\overline{\mathcal{C}} : \mathrm{Cat}, \qquad \text{(With strict category laws)}$$
$$i : \mathcal{C} \to \overline{\mathcal{C}}, \qquad \text{(With strict functor laws)}$$
$$\text{(Homogeneous fibrant replacement for the components of } \overline{\mathcal{C}}\text{)},$$
$$\text{(Saturation for } \overline{\mathcal{C}}\text{)}.$$

Not clear why $i : \mathcal{C} \to \overline{\mathcal{C}}$ would be a weak equivalence.

Propositional truncation without homogeneous fibrant replacement

Remark by Cherubini, Coquand, and Hutzler (2023): the propositional truncation can be defined without homogeneous fibrant replacement.

$$
\begin{aligned}
&\| X \| : \mathrm{Set}, \\
&i : X \to \| X \|, \\
&\mathsf{ext} : (x : \| X \|) \to \mathsf{HasExt}(\| X \|).
\end{aligned}
$$

$\rightsquigarrow$ The fibrancy of $\| X \|$ can be proven.

This can be seen as a strict Rezk completion corresponding to a model structure on **Set** that presents propositions.

Strict Rezk completion of categories

$$
\begin{aligned}
&\overline{\mathcal{C}} : \mathbf{Cat}, \\
&i : \mathcal{C} \to \overline{\mathcal{C}}, \\
&\mathrm{ext}_{\mathrm{Ob}} : (x \in \overline{\mathcal{C}}) \to \mathrm{HasExt}((y \in \overline{\mathcal{C}}) \times (x \cong y)), \quad \text{(isomorphism extension structure)} \\
&\mathrm{ext}_{\mathrm{Hom}} : ((f : x \to y) \in \overline{\mathcal{C}}) \to \mathrm{HasExt}((g : x \to y) \times (f = g)), \\
&\mathrm{ext}_{\mathrm{EqHom}} : (p : f = g) \to \mathrm{HasExt}((q : f = g) \times \top). \quad \text{(redundant)}
\end{aligned}
$$

Strict Rezk completion of models of HoTT

$$\overline{\mathcal{M}} : \mathbf{Mod}_{\mathsf{HoTT}},$$

$$i : \mathcal{M} \to \overline{\mathcal{M}},$$

$$\mathsf{ext}_{\mathsf{Tm}} : (x : \overline{\mathcal{M}}.\mathsf{Tm}(\Gamma, A))$$

$$\quad \to \mathsf{HasExt}((y : \overline{\mathcal{M}}.\mathsf{Tm}(\Gamma, A)) \times (p : \overline{\mathcal{M}}.\mathsf{Tm}(\Gamma, \mathsf{Id}_A(x, y)))),$$

$$\mathsf{ext}_{\mathsf{Ty}} : (A : \overline{\mathcal{M}}.\mathsf{Ty}(\Gamma))$$

(identification extension structure)

$$\quad \to \mathsf{HasExt}((B : \overline{\mathcal{M}}.\mathsf{Ty}(\Gamma)) \times (A \simeq B)).$$

(redundant: $\mathsf{Ty}(\Gamma) \cong \mathsf{Tm}(\Gamma, \mathcal{U}) + \mathsf{UA}$)

Equivalence extension structure

We have to prove fibrancy of the components of $\overline{\mathcal{C}}$, $\overline{\mathcal{M}}$.

Fibrancy of the universe $\mathrm{Set}_{\mathsf{fib}}$ follows from the equivalence extension structure

$$(A : \mathrm{Set}_{\mathsf{fib}}) \to \mathsf{HasExt}((B : \mathrm{Set}_{\mathsf{fib}}) \times (A \simeq B)).$$

Fibrancy from reflexive graphs

Lemma (Sattler)

Let $A = (V_A, E_A)$ be a global reflexive graph.

$$V_A \xrightarrow{r} E_A \overset{\pi_1}{\underset{\pi_2}{\rightrightarrows}} V_A$$

If A is **homotopical** *and has* **coercion**, *then V_A is fibrant.*

$$\mathrm{com}_{V_A}^{r \to s}(b, t) \triangleq \mathrm{ext}(b, [\alpha \mapsto (t(s), \mathrm{coe}^{r \to s}(t)), (r = s) \mapsto (b, r(b))]).1$$

Fibrancy from reflexive graphs

A reflexive graph $V_A : \mathrm{Set}, E_A : V_A \times V_A \to \mathrm{Set}$

$$V_A \xrightarrow{r} E_A \underset{\pi_2}{\overset{\pi_1}{\rightrightarrows}} V_A$$

is **homotopical** (is a **path object**) if we have

$$\mathsf{ext} : (x : V_A) \to \mathsf{HasExt}((y : V_A) \times E_A(x, y)).$$

(Equivalently: $\pi_1 : E_A \to V_A$ is a trivial fibration.)

Fibrancy from reflexive graphs

A reflexive graph $V_A : \mathrm{Set}, E_A : V_A \times V_A \to \mathrm{Set}$

$$V_A \xrightarrow{r} E_A \underset{\pi_2}{\overset{\pi_1}{\rightrightarrows}} V_A$$

has **coercion** if we have

$$\text{coe} : (a : \mathbb{I} \to V_A)(r, s : \mathbb{I}) \to E_A(a(r), a(s)).$$

s.t. $\text{coe}^{r \to r}(a) = r(a)$.

Fibrancy from reflexive graphs: the universe

Apply the lemma to the reflexive graph

$$V_A = \mathrm{Set}_{\mathsf{fib}},$$
$$E_A(X, Y) = \mathrm{Equiv}(X, Y)$$

Homotopicality is the equivalence extension property (Glue-types).

The coercion operation is constructed from the fibrancy of the elements of $\mathrm{Set}_{\mathsf{fib}}$.

Fibrancy from reflexive graphs: generalization

Lemma

Let $B = (V_B, E_B)$ be a global displayed reflexive graph over a base $A = (V_A, E_A)$.

$$\begin{array}{ccccc} V_B & \xrightarrow{r} & E_B & \overset{\pi_1}{\underset{\pi_2}{\rightrightarrows}} & V_B \\ \downarrow & & \downarrow & & \downarrow \\ V_A & \xrightarrow{r} & E_A & \overset{\pi_1}{\underset{\pi_2}{\rightrightarrows}} & V_A \end{array}$$

If B is **homotopical** *over A ($E_B \to E_A \times_{V_A} V_B$ is a trivial fibration) and both A and B have compatible* **coercion** *operations, then $V_B \to V_A$ is a fibration.*

Path functors

Categories have a “cylindrical” model structure. (Williamson 2016)

$=$ Constructed using right adjoint path functors

$$\mathcal{C} \xrightarrow{r} \mathrm{Path}(\mathcal{C}) \underset{\pi_2}{\overset{\pi_1}{\rightrightarrows}} \mathcal{C}$$

$\mathrm{Path}(\mathcal{C}) = \mathcal{C}^{\cong}$

Fibrancy from reflexive graphs for components of categories

$$\begin{array}{ccccc}
\mathbf{Cat}(\mathcal{B},\overline{\mathcal{X}}) & \xrightarrow{r} & \mathbf{Cat}(\mathcal{B},\mathrm{Path}(\overline{\mathcal{X}})) & \overset{\pi_1}{\underset{\pi_2}{\rightrightarrows}} & \mathbf{Cat}(\mathcal{B},\overline{\mathcal{X}}) \\
\downarrow & & \downarrow & & \downarrow \\
\mathbf{Cat}(\mathcal{A},\overline{\mathcal{X}}) & \xrightarrow{r} & \mathbf{Cat}(\mathcal{A},\mathrm{Path}(\overline{\mathcal{X}})) & \overset{\pi_1}{\underset{\pi_2}{\rightrightarrows}} & \mathbf{Cat}(\mathcal{A},\overline{\mathcal{X}})
\end{array}$$

for any generating cofibration $i : \mathcal{A} \to \mathcal{B}$ in

$$I = \{\ \{\} \to \{x\}, \{x, y\} \to \{x \to y\}, \{x \rightrightarrows y\} \to \{x \to y\}\}.$$

Homotopicality follow from the isomorphism extension structure.

Coercion operations are defined using the universal property of $\overline{\mathcal{C}}^{\mathbb{I}}$. ($\overline{\mathcal{C}}^{\mathbb{I}} \cong \overline{\mathcal{C}^{\mathbb{I}}}$)

Pseudo-path functors

Models of HoTT should[2] have a "pseudo-cylindrical" weak model structure.

= Constructed by right adjoint path and reflexive-loop functors

$$\mathrm{ReflLoop}(\mathcal{M}) \xrightarrow{\pi_e} \mathrm{Path}(\mathcal{M}) \overset{\pi_1}{\underset{\pi_2}{\rightrightarrows}} \mathcal{M}$$

(Called weak Quillen cylinders by Henry (2023))

[2] I have only checked what was needed for the constructions,
+ We need a notion of algebraic weak model structure.

Fibrancy from pseudo-reflexive graphs

Generalize to pseudo-reflexive graphs:

$$\begin{array}{ccccc}
\mathbf{Mod}_{\mathsf{HoTT}}(\mathcal{B}, \mathsf{ReflLoop}(\overline{\mathcal{X}})) & \xrightarrow{\pi_e} & \mathbf{Mod}_{\mathsf{HoTT}}(\mathcal{B}, \mathsf{Path}(\overline{\mathcal{X}})) & \overset{\pi_1}{\underset{\pi_2}{\rightrightarrows}} & \mathbf{Mod}_{\mathsf{HoTT}}(\mathcal{B}, \overline{\mathcal{X}}) \\
\downarrow & & \downarrow & & \downarrow \\
\mathbf{Mod}_{\mathsf{HoTT}}(\mathcal{A}, \mathsf{ReflLoop}(\overline{\mathcal{X}})) & \xrightarrow{\pi_e} & \mathbf{Mod}_{\mathsf{HoTT}}(\mathcal{A}, \mathsf{Path}(\overline{\mathcal{X}})) & \overset{\pi_1}{\underset{\pi_2}{\rightrightarrows}} & \mathbf{Mod}_{\mathsf{HoTT}}(\mathcal{A}, \overline{\mathcal{X}})
\end{array}$$

for any generating cofibration $i : \mathcal{A} \to \mathcal{B}$ in

$$I = \{\mathbf{0}[\mathbf{\Gamma} \vdash \boldsymbol{A} \text{ type}] \to \mathbf{0}[\mathbf{\Gamma} \vdash \boldsymbol{a} : \boldsymbol{A}]\}.$$

Conclusion

Paper: [arXiv:2311.05849]

- Application: the strict Rezk completion of the syntax of HoTT can be used to prove homotopy canonicity for HoTT.
- The strict Rezk completion is a fibrant replacement in $[\square^{op}, \mathbf{Mod}_{\mathsf{HoTT}}]$.
  $\rightsquigarrow$ Weak model structure on $[\square^{op}, \mathbf{Mod}_{\mathsf{HoTT}}]$?
  Comparison with the weak model structure on $\mathbf{Mod}_{\mathsf{HoTT}}$?
- Can we add stability of the extension structures under substitution.

$$\mathsf{ext}_{\mathsf{Tm}}(\Gamma, A, a, \alpha \mapsto (b, e))[f] = \mathsf{ext}_{\mathsf{Tm}}(\Delta, A[f], a[f], \alpha \mapsto (b[f], e[f])).$$

Stability under renamings is needed for applications to homotopy normalization.

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