A 2-CATEGORICAL PROOF OF FROBENIUS FOR FIBRATIONS DEFINED FROM A GENERIC POINT

SINA HAZRATPOUR & EMILY RIEHL

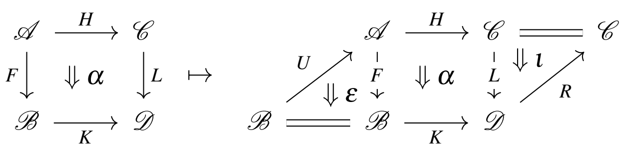
The "Frobenius condition" is a property of weak factorization systems (WFS) that requires pullback along morphisms in the right class to preserve morphisms in the left class ([GS]). In a locally cartesian closed category with a WFS, this condition is logically equivalent to the condition that the morphisms in the right class are closed under pushforwards along morphisms in the right class. In a full model category, this corresponds to right properness. The Frobenius condition is instrumental in obtaining models of Homotopy Type Theory (HoTT) in simplicial and cubical sets as it gives the interpretation of dependent product (Π) types.

Voevodsky's simplicial model of HoTT ([KLV]) justifies the Frobenius condition through the non-constructive use of minimal fibrations. Coquand gave a constructive proof of the Frobenius condition in Cubical Type Theory. Coquand's proof takes advantage of an important lemma which reduces fibration structures to the more manageable composition structures ([C], [ABCFHL]).

However, this proof is difficult to understand for the category theorists and homotopy theorists who are not well-versed in cubical type theory, not least due to its heavy syntax. Several experts in the field (Awodey, Gambino, Sattler) worked on obtaining a categorical proof which is more conceptual and less syntactical ([A]). Building on their work, we give a complete proof of an enhanced version of the Frobenius condition, namely the *functorial* Frobenius for *structured fibrations* defined from a generic point in a locally cartesian closed category ([HR]). Our proof does not require connection structures on the interval object since we work with the "unbiased" fibrations (due to Awodey). We verified, through the use of internal language, that our 2-categorical proof corresponds to Coquand's type-theoretic proof.

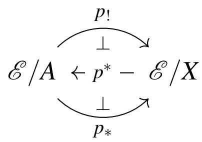
Our proof is novel in that it deploys 2-category theory, specifically the calculus of mates, both to define the natural maps in the diagrams that we deploy and to prove their commutativity. The use of 2-categorical machinery instead of reasoning by universal properties leads to an equational approach which makes our diagram-chasing proof semi-automatic and the translation to type theory more transparent.

The mates correspondence gives an extended, double-categorical, version of adjoint transposition ([KS]): a suitably-oriented 2-cell in a square involving parallel left adjoints is mates with another 2-cell in the corresponding square formed by their right adjoints.



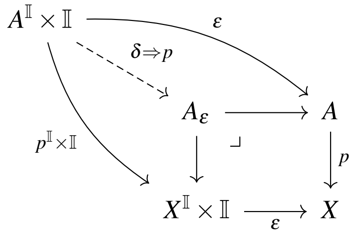
Importantly, the mates correspondence is "natural" in a double categorical sense, which means in practice to prove the commutativity of a diagram involving such 2-cells in can be expeditious to pass instead to their mates and establish commutativity there.

We express our diagrams using the double categorical framework which we deploy to great effect to manipulate natural transformations between the adjoint triples: In a locally cartesian closed category $\mathscr{E}$, every morphism $p\colon A \to X$ gives rise to an adjoint triple $p_! \dashv p^* \dashv p_*$ between the slice categories over A and over X:



In the internal type theory of $\mathscr{E}$, the map $p\colon A \to X$ is seen as a type A in the context X, the left adjoint $p_!$ (post-composition with p) corresponds to the operation of forming the dependent sum type $\sum_{(x:X)} A(x)$, and p^* (pulling back along p) to the substitution along p, and p_* corresponds to the operation of forming the dependent product type $\prod_{(x:X)} A(x)$. Frobenius condition states that if the map p carries a fibration structure, then for any other map $q\colon B \to A$ with a fibration structure, the pushforward $p_*(q)$, interpreting the Π-type, also carries a fibration structure which we construct by applying the mate calculus to the units and counits produced by the triple adjunction above.

Fibrations are defined from a class of trivial fibrations with few closure properties. We define a map $p\colon A \to X$ to be a *fibration* just when the induced map to the pullback in the naturality square for the evaluation transformation is a trivial fibration:



Here $\mathbb{I}$ is an object of the category $\mathscr{E}$, and thought of as some sort of "interval"—though in our present context no interval structure is required.

We enhance the definition above to the notion of *structured fibrations* building on the *fibred notion of structure* from [Sw]. We prove that under the natural functorial enhancements of our hypotheses on the class of trivial fibrations, that the corresponding structured fibrations admit a *functorial Frobenius operator* in the sense of Gambino–Sattler [GS]. Moreover, we demonstrate that the induced fibration structure on the pushforward is stable under substitution under natural conditions.

References

[ABCFHL] Carlo Angiuli, Guillaume Brunerie, Thierry Coquand, Robert Harper, Kuen-Bang Hou (Favonia), Daniel R. Licata, Syntax and models of Cartesian cubical type theory, *Math. Struct. Comput. Sci.*, **31(4)**, 424–468, 2021.

[A] Steve Awodey, Cartesian cubical model categories, 2023, arXiv:2305.00893

[C] Thierry Coquand, Variation on Cubical sets, 2014, `https://www.cse.chalmers.se/~coquand/diag.pdf`

[GS] Nicola Gambino and Christian Sattler, The Frobenius condition, right properness, and uniform fibrations, *J. Pure Appl. Algebra*, **221 (12)**, 3027–3068, 2017.

[HR] Sina Hazratpour and Emily Riehl. A 2-Categorical Proof of Frobenius for Fibrations Defined From a Generic Point. Mathematical Structures in Computer Science (to appear), arxiv:1808.08291

[KLV] Chris Kapulkin and Peter LeFanu Lumsdaine. The simplicial model of univalent foundations (after Voevodsky). *J. Eur. Math. Soc.*, **23 (6)** , 2071–2126, 2021, arXiv:1211.2851.

[KS] Kelly, G. M. & Street, Ross. (1974). Review of the elements of 2-categories, *Category Seminar Proc. Sem., Sydney, 1972/1973, Lecture Notes in Math.*, **420**, 75–103.

[N] Clive Newstead, Algebraic models of dependent type theory, PhD Thesis, Carnegie Mellon University, 2018.

[P] Duško Pavlović, Categorical interpolation: Descent and the Beck-Chevalley condition without direct images, in: *Category Theory: Proceedings of the International Conference held in Como, Italy, July 22-28, 1990*, Lecture Notes in Mathematics **1488**, 306–325, 1991.

[Sh] Michael Shulman, All $(\infty, 1)$-toposes have strict univalent universes, 2019, arXiv:1904.07004

[Sw] Andrew W. Swan, Definable and Non-definable Notions of Structure, 2022, arXiv:2206.13643