Ordinal exponentiation in homotopy type theory

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Ordinals are a powerful tool for establishing consistency of logical theories, proving termination of processes and justifying induction and recursion. In homotopy type theory, a standard definition of an ordinal (cf. [10, §10.3]) is a type equipped with an order relation that is transitive, extensional (elements with the same predecessors are equal) and wellfounded (the order admits the principle of transfinite induction). In previous work, we have considered logical and order-theoretic properties of type-theoretic ordinals [5, 6], as well as their relationship with their set-theoretic counterpart [2]. In this work, we are interested in their algebraic properties. It is well known classically that the ordinals admit a rich theory of arithmetic, with operations of addition, multiplication and exponentiation which extend the usual arithmetic for the natural numbers. These arithmetic operations and their theory often play a crucial role in the application of ordinals, e.g. for constructing an appropriate measure witnessing termination. Can ordinal arithmetic also be developed constructively in homotopy type theory?

Ordinal exponentiation via case distinction Addition and multiplication can be realised by disjoint union and Cartesian product of the underlying types of the ordinals, respectively. Their basic properties were investigated by Escardó [4].

The case of exponentiation is constructively more challenging. Most classical textbook accounts define exponentiation α^β by classifiability induction on β, i.e., by a case distinction on whether β is zero, a successor, or a limit ordinal:

$$\alpha^0 = 1 \qquad \alpha^{\beta+1} = \alpha^\beta \times \alpha \qquad \alpha^{\lim_{i:I} f(i)} = \lim_{i:I} \alpha^{f(i)} \quad (\text{for } \alpha \neq 0) \qquad (\dagger)$$

Unfortunately, the ability to make this case distinction on arbitrary ordinals is equivalent to excluded middle [6, Thm. 63], and is thus not available constructively. However, the equations in (†) do form a good *specification* of exponentiation.

Ordinal exponentiation as functions with finite support Sierpiński [9, §XIV.15] gives an explicit construction of the exponential α^β as the collection of functions $\beta \to \alpha$ with *finite support*, i.e., functions $f : \beta \to \{x \mid 0 \leq x < \alpha\}$ such that $f(x) > 0$ for only finitely many x.

While this definition works well classically, the order relation it induces does not seem to be well-behaved constructively. The usual classical argument that the exponential is an ordinal requires decidability of the order on α, and decidable equality on α seems to be required to verify the expected properties (such as the specification (†)) of this ordinal. In general, neither of these assumptions are constructively justified.

Ordinal exponentiation, constructively For ordinals α with a least element, i.e., for ordinals of the form $\alpha = 1 + \gamma$ for some ordinal γ, we are able to define the exponential α^β constructively, by considering an intensional variant of Sierpiński's construction.

Since β is an ordinal, we can think of a finitely supported function from β to γ as a finite list of output-input[1] pairs $[(c_0, b_0), (c_1, b_1), \ldots, (c_n, b_n)] : \mathsf{List}\,(\gamma \times \beta)$ which is strictly decreasing in the second argument (to enforce that each input has a unique output), with all inputs not occurring in the list being sent to the least element. Write

$$\mathsf{D_2List}(\gamma, \beta) :\equiv \big(\Sigma\,\ell : \mathsf{List}\,(\gamma \times \beta)\big)\ \mathsf{is\text{-}decreasing}\,(\mathsf{map}\,\pi_2\,\ell)$$

for the type of such lists of pairs decreasing in the second component. The idea of the intensional presentation using lists is similar to—but more general than—Setzer's sketch [8, App. A] of the construction of exponentials with base ω.

Theorem. *The type $\mathsf{D_2List}(\gamma, \beta)$ is an ordinal, when ordered lexicographically. Moreover, it satisfies the specification (†) for $\alpha = 1 + \gamma$.*

Constructively, this is the most we can hope for: having an exponential α^β which satisfies the specification (†) for *all* α is equivalent to excluded middle, if limits are defined as suprema of increasing sequences. More generally, excluded middle follows as soon as there is an operation $\mathsf{exp} : \mathsf{Ord} \to \mathsf{Ord} \to \mathsf{Ord}$ which satisfies the first two equations in the specification (†), and which is monotone in its second argument.

We note that Coquand, Lombardi and Neuwirth [1] develop a notion of constructive ordinal similar to one proposed by Martin-Löf [7, Chapter 3]. This notion of ordinal allows for a definition of the exponential α^β by induction on β. We leave a detailed comparison of this approach to ordinal arithmetic and our approach for future work.

Formalisation We have formalised our results in Agda, building on Escardó's TypeTopology development [3]. We have found Agda extremely valuable in developing our proofs as the intensional nature of our construction makes for rather combinatorial arguments. The source code can be found at `https://github.com/fredrikNordvallForsberg/TypeTopology/blob/exponentiation/source/Ordinals/Exponentiation.lagda`.

References

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[1] We prefer to use output-input pairs rather than input-output pairs so that their order corresponds to the usual order on the product $\gamma \times \beta$, which is reverse lexicographic.

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