Connected Covers in Cubical Agda

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Connected covers were originally introduced in 1952 by Cartan and Serre [4, 5] and, independently, Whitehead [13]. Using connectivity we can analyze homotopy groups using (co)homology [2, 6]. Studying the homotopy groups of important spaces (especially the spheres) has been central to synthetic homotopy theory from the beginning [c.f. 12, Chapter 8], so it is useful to understand and formalize key properties of connected covers (defined and motivated below) in a range of settings. Work on connected covers in HoTT has been done previously. In particular, a proof of the universal property of connected covers (stated below) appeara in Buchholtz, van Doorn, and Rijke [3] and was formalized by those authors in Lean 2 [8] and also, as part of the work of Christensen and Scoccola [6], in Coq [1]. Other relevant work includes that of Shulman and Hou (Favonia) [11] – who work with the universal cover of a space, which is another name for the 1st connected cover. The work being presented here is a formalization of some of the theory of connected covers in Cubical Agda.

If X is a pointed space, we write $X\langle n\rangle$ for the nth connected cover of X – defined to be the fiber of the truncation map $|-|_n : X \to \|X\|_n$. The nth connected cover is n-connected and there is a universal map $X\langle n\rangle \to X$ – in the sense that if Y is any n-connected, pointed space, then we have an equivalence of pointed function types: $(Y \to_\bullet X) \simeq (Y \to_\bullet X\langle n\rangle)$ given by composition with the universal map. Together these imply the basic facts that $\pi_k(X\langle n\rangle) = 0$ if $k \leq n$ and $\pi_k(X\langle n\rangle) = \pi_k X$ otherwise. Another basic fact about connected covers is that $(X\langle n\rangle)\langle n+1\rangle = X\langle n+1\rangle$ so there is a universal map $X\langle n+1\rangle \to X\langle n\rangle$. The fiber of this map is $K(\pi_n X, n+1)$ – the main result of this work is a formal proof of this fact. The proof uses Whitehead's lemma, which states that if X and Y are spaces with finite h-levels, then $f : X \to Y$ is an equivalence if and only if $\|f\|_0 : \|X\|_0 \to \|Y\|_0$ is an equivalence and $\pi_n(f) : \pi_n(X, x) \to \pi_n(Y, f(x))$ is an equivalence for each $n \geq 1$ and each $x : X$. Whitehead's lemma was also formalized in Cubical Agda as a part of this work [7].

To get a feeling for how connected covers can be useful for studying homotopy groups, consider the following simple example (with details omitted): The wedge product of a family of spaces (denoted with a $\vee$ or a $\bigvee$ depending on the size of the family) is the space that results from "gluing" those spaces together at their basepoints. The 1st connected cover of $\mathbb{S}^1 \vee \mathbb{S}^2$ is $\bigvee_{\mathbb{Z}} \mathbb{S}^2$. It is a well-known fact in classical algebraic topology that the nth homology group of a wedge sum of a family of spaces is the direct product of the nth homology groups of those spaces [see e.g. 9, Cor. 2.25]. So, using the Hurewicz theorem [6] – which tells us that the $n+1$th homotopy group of an n connected space is equal to the $n+1$th homology group – and some facts we mentioned above, we have: $\pi_2(\mathbb{S}^1 \vee \mathbb{S}^2) = \pi_2(\bigvee_{\mathbb{Z}} \mathbb{S}^2) = H_2(\bigvee_{\mathbb{Z}} \mathbb{S}^2) = \bigoplus_{\mathbb{Z}} H_2(\mathbb{S}^2) = \bigoplus_{\mathbb{Z}} \mathbb{Z}$.

The code from the formalization is available online [10].

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