A formal treatment of univalent completions

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1 Introduction

Internal to HoTT/UF, category theory often comes in two flavors: set-category (or strict-category) theory, where one deals with categories whose type of objects is a (h)set; or, univalent category theory, where one deals with categories with a structure-identity principle for isomorphic objects. Univalent category theory is often preferred for multiple reasons. First, there are not many set-categories. Second, univalent categories have the desired type of equivalences: a fully faithful and (mere) essentially surjective functor, between univalent categories, is an equivalence (even an isomorphism/identity) of categories. More generally, universal constructions internal to univalent categories are necessarily unique (up to identity). Furthermore, in the interpretation of HoTT/UF in the simplicial set model, categories correspond to Segal spaces, while univalent categories correspond to complete Segal spaces, up to truncation.

Of course, some categories are not univalent. Furthermore, solutions of universal constructions, such as Kleisli categories, are not necessarily univalent. Fortunately, Ahrens et al. provide a process to make an arbitrary category into a univalent one freely; referred to as the *Rezk completion*. Furthermore, it is shown how the theory of univalent categories and the Rezk completion generalizes to monoidal categories [4], enriched categories [3], etcetera. In this project, we aim to generalize the theory internal to (suitably structured) bicategories.

2 Rezk completions of (structured) categories

In [1], it is shown how every category is *weakly equivalent* to a univalent one. That is, every category $\mathcal{C}$ admits a (necessarily unique) univalent category $\mathsf{RC}(\mathcal{C})$ together with a functor $\eta_{\mathcal{C}} : \mathcal{C} \to \mathsf{RC}(\mathcal{C})$ which is fully faithful and (mere) essentially surjective. The pair consisting of $\mathsf{RC}(\mathcal{C})$ and $\eta_{\mathcal{C}}$ is called the *Rezk completion*. Furthermore, the Rezk completion necessarily satisfies the universal property of the *free univalent completion*:

Theorem 1. *(thm 8.4, [1]) Let $F : \mathcal{C} \to \mathcal{D}$ be a functor with $\mathcal{D}$ univalent. Assume F is fully faithful and (mere) essentially surjective. Then, for any univalent category $\mathcal{E}$, the precomposition functor $(F \cdot -) : [\mathcal{D}, \mathcal{E}] \to [\mathcal{C}, \mathcal{E}]$, is an isomorphism of categories.*

In particular, this means that every functor from $\mathcal{C}$ into a univalent category $\mathcal{D}$ factors uniquely through $\eta_{\mathcal{C}}$.

Furthermore, we have the following *concrete* construction of the Rezk completion:

Theorem 2. *(thm 8.5, [1]) Let $\mathcal{C}$ be a small category and $よ_{\mathcal{C}} : \mathcal{C} \to [\mathcal{C}^{\mathsf{op}}, \mathsf{hSet}]$ its Yoneda embedding. Then, the (replete) image of $よ_{\mathcal{C}}$ is univalent and the image projection is a weak equivalence.*

Theorems 1 and 2 suitably generalize to monoidal, resp. enriched, categories, see [4][3].

3 Our work: Rezk completions internal to bicategories

In this section, we describe the goal (and current status) of our project. We aim to generalize theorem 1 and theorem 2 internal to a (suitably structured) bicategory. In particular, we want to rediscover the analogous results for monoidal categories and enriched categories. Furthermore, we can instantiate the results to concrete bicategories of *structured* categories, such as: (co)complete categories, closed categories, abelian categories, fibered categories, and so on.

The main ingredient we seem to need is a *Yoneda structure* on a bicategory [2] generalizing the (enriched) category of (enriched) presheaves and the (enriched) Yoneda embedding. For ease of reading, we ignore size issues.

A **Yoneda structure** on a bicategory B assigns to every object $X : \mathsf{B}$, an *object* $\mathcal{P}X$ *of presheaves* and a *Yoneda morphism* $よ_X : X \to \mathcal{P}X$. In the bicategory of (enriched) categories, functors and natural transformations, the object of presheaves is given by the (enriched) category of (enriched) presheaves and a Yoneda morphism is the (enriched) Yoneda embedding.

A Yoneda structure provides sufficient structure to suitably interpret many constructions arising in enriched category theory, such as enriched fully faithful functors; leading to the definition of **fully faithful morphisms** (see p. 360, section 3, [2]).

The interpretation of *essentially surjectiveness* is more subtle. A (mere) essentially surjective (enriched) functor is necessarily left orthogonal to any (enriched) fully faithful functor whose source is a univalent (enriched) category. Hence, the definition of fully faithful morphisms can be defined relative to the Yoneda structure. However, we need a predicate corresponding to the requirement that the source is univalent. Let B be a bicategory and $\mathsf{B}_{\mathbb{P}}$ a full subbicategory classified by a predicate $\mathbb{P} : \mathsf{ob}(\mathsf{B}) \to \mathsf{hProp}$. Furthermore, we assume that $\mathsf{B}_{\mathbb{P}}$ is a globally univalent bicategory, and for any $Z : \mathsf{B}_{\mathbb{P}}$ satisfying $\mathbb{P}$, the (hom-)category $\mathsf{B}(X, Z)$ is univalent. Hence, $\mathsf{B}_{\mathbb{P}}$ is a univalent full subbicategory of B. Finally, we define a morphism to be **eso** if it is left orthogonal to the fully faithful morphisms whose source satisfies $\mathbb{P}$.

Conjecture 3. Let $f : X \to Y$ be a fully faithful morphism and an eso morphism with Y satisfying $\mathbb{P}$. Then, for any $Z : \mathsf{B}_{\mathbb{P}}$ the precomposition functor $\mathsf{B}(f, Z) : \mathsf{B}(Y, Z) \to \mathsf{B}(X, Z)$, is an equivalence of (hom-)categories.

Conjecture 3 implies that (the image of) an (eso, ff)-factorization of the Yoneda embedding is necessarily the $\mathbb{P}$-completion:

Corollary 4. Let $X : \mathsf{B}$ be an object. Assume the Yoneda morphism $よ_X : X \to \mathcal{P}X$ factors as $X \xrightarrow{\eta_X} \hat{X} \xrightarrow{\iota_X} \mathcal{P}X$; where η_X is an eso morphism, ι_X is a fully faithful morphism, and $\hat{X}$ satisfies $\mathbb{P}$. Then, η_X is a universal arrow for the inclusion of $\mathsf{B}_{\mathbb{P}}$ into B.

References

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