Primitive Recursive (Homotopy) Type Theory

Ulrik Buchholtz[1] and Johannes Schipp von Branitz

School of Computer Science, University of Nottingham

We show that restricting the elimination principle of the natural numbers type in Martin-Löf Type Theory (MLTT) to a universe of types not containing Π-types ensures that all definable functions are primitive recursive. This extends the concept of primitive recursiveness to general types. We discuss extensions to univalent type theories and other notions of computability. We are inspired by earlier work by Martin Hofmann [2], work on Joyal's arithmetic universes [3], and Hugo Herbelin and Ludovic Patey's sketched Calculus of Primitive Recursive Constructions [1].

We define a theory T_{pr} that is a subtheory of MLTT with two universes $\mathrm{U}_0 : \mathrm{U}_1$, such that all inductive types are finitary and U_0 is restricted to not contain Π-types:

$$\frac{\vdash A : \mathrm{U}_\alpha \qquad a : A \vdash B(a) : \mathrm{U}_\alpha}{\vdash (a : A) \to B(a) : \mathrm{U}_{\min(1,\alpha)}}$$

We prove soundness such that all functions $\mathbb{N} \to \mathbb{N}$ are primitive recursive. We also adopt synthetic Tait computability [4] to easily prove canonicity.

Semantics The theory T_{pr} is a subtheory of MLTT. As such, any model for MLTT is a model for T_{pr}. This includes any presheaf topos on a small category. In particular, we have a standard model

$$[\![-]\!]_{\mathrm{Set}} : \mathrm{T}_{\mathrm{pr}} \to \mathrm{Set}$$

in the topos of sets. This model maps the syntactic natural numbers N to the actual natural numbers $\mathbb{N}$.

We prove soundness by giving an interpretation $[\![-]\!]_{\mathcal{R}}$ for T_{pr} in a certain sheaf topos $\mathcal{R}$ of primitive recursive functions.

This topos has the property that morphisms $\mathcal{R}([\![\mathrm{N}]\!]_{\mathcal{R}}, [\![\mathrm{N}]\!]_{\mathcal{R}})$ are exactly primitive recursive functions $\mathbb{N} \to \mathbb{N}$. We use a gluing argument to show that functions $\mathrm{Set}([\![\mathrm{N}]\!]_{\mathrm{Set}}, [\![\mathrm{N}]\!]_{\mathrm{Set}})$ are primitive recursive.

We define the category R of primitive recursive functions: Its only two objects $\mathbf{1}$ and $\mathbb{N}$ are related by morphisms $\mathrm{R}(\mathbb{N}, \mathbb{N})$, the primitive recursive functions of type $\mathbb{N} \to \mathbb{N}$, and $\mathrm{R}(\mathbf{1}, \mathbb{N}) := \mathbb{N}$. The object $\mathbf{1}$ is terminal. We sometimes write $\mathbb{N}^0 := \mathbf{1}$. Note that R has finite products, so it is equivalent to the Lawvere theory of primitive recursive functions.

Let J_{fin} be the Grothendieck topology on R generated by the basis consisting of finite jointly surjective families. We denote the sheaf topos as $\mathcal{R} := \mathrm{Sh}(\mathrm{R}, J_{\mathrm{fin}})$. The topology J_{fin} is subcanonical, so $\mathcal{R}(\mathbf{a}\mathrm{y}_{\mathbb{N}}, \mathbf{a}\mathrm{y}_{\mathbb{N}}) \cong \mathrm{R}(\mathbb{N}, \mathbb{N})$. Just like any Grothendieck topos, $\mathcal{R}$ has a natural numbers object $\mathbb{N}$. However, we note that N_0 can be soundly interpreted as $\mathrm{y}_{\mathbb{N}}$. Then the Yoneda lemma implies that natural transformations $\mathcal{R}(\mathrm{y}_{\mathbb{N}}, \mathrm{y}_{\mathbb{N}})$ are primitive recursive functions $\mathbb{N} \to \mathbb{N}$. The interpretation of the other type-theoretic constructors and rules is standard.

Theorem 1. *There is a sound interpretation* $[\![-]\!]_{\mathcal{R}} : \mathrm{T}_{\mathrm{pr}} \to \mathcal{R}$ *satisfying* $[\![\mathrm{N}_0]\!]_{\mathcal{R}} = \mathrm{y}_{\mathbb{N}}$.

We assume strong cumulative universes $\mathcal{U}_0 < \mathcal{U}_1$ in $\mathcal{R}$ and define the universes $\mathcal{U}_0^{\mathrm{pr}}, \hat{\mathcal{U}}_0^{\mathrm{pr}}$, respectively. We have the type

$$\mathcal{U}_0^{\mathrm{pr}} := (X : \mathcal{U}_0) \times \Big((g : X) \to (h : \mathrm{y}_{\mathbb{N}} \to X \to X) \to \big((f : \mathrm{y}_{\mathbb{N}} \to X) \times \mathrm{comp}(f, g, h)\big)\Big)$$

with

$$\mathrm{comp}(f, g, h) := (f(\mathrm{zero}) = g) \times ((n : \mathrm{y}_{\mathbb{N}}) \to f(\mathrm{succ}(n)) = h(n, fn)).$$

It consists of the $\mathcal{U}_0$-small types which N_0 can eliminate into.

We also have the object

$$\hat{\mathcal{U}}_0^{\mathrm{pr}} := (X : \mathcal{U}_0) \times (r : \mathrm{y}_{\mathbb{N}} \to (X + \mathbf{1})) \times (s : (X + \mathbf{1}) \to \mathrm{y}_{\mathbb{N}}) \times (r \circ s = \mathrm{id}_{X+\mathbf{1}})$$

of types X together with a retraction $\mathrm{y}_{\mathbb{N}} \to X + \mathbf{1}$.

It doesn't seem possible to prove internally that $\mathcal{U}_0^{\mathrm{pr}}$ is closed under Σ-types, but for $\hat{\mathcal{U}}_0^{\mathrm{pr}}$ it is.

The reason we work in the topos $\mathcal{R}$ rather than PrR is that $\mathbf{2}$ is not a retract of $\mathrm{y}_{\mathbb{N}}$ in PrR. Thus, the restriction to sheaves is necessary to prove:

Lemma 2. *All finite types are in* $\hat{\mathcal{U}}_0^{\mathrm{pr}}$*.*

Since the representable sheaf $\mathrm{y}_{\mathbb{N}}$ is in $\mathcal{U}_0^{\mathrm{pr}}$ and has decidable equality in $\mathcal{R}$, we get a map

$$\Phi : \hat{\mathcal{U}}_0^{\mathrm{pr}} \to \mathcal{U}_0^{\mathrm{pr}}$$

preserving underlying types.

Lemma 3. *The universe* $\hat{\mathcal{U}}_0^{\mathrm{pr}}$ *is closed under* Σ*-types.*

The interpretations $[\![-]\!]_{\mathrm{Set}}$ and $[\![-]\!]_{\mathcal{R}}$ are flat functors. They are even locally cartesian closed, because they define models of a theory expressed as an LCCC. Hence, they extend to left exact functors $\widehat{[\![-]\!]_{\mathcal{R}}}$ and $\widehat{[\![-]\!]_{\mathrm{Set}}}$ along their Yoneda embedding. It follows that we have a left exact extension

$$\rho := \Gamma(\widehat{[\![-]\!]_{\mathcal{R}}}) \times \widehat{[\![-]\!]_{\mathrm{Set}}} : \Pr \mathrm{T}_{\mathrm{pr}} \to \mathrm{Set}$$

and the comma category $\mathcal{G} := \mathrm{Set} \downarrow \rho$ is a logos. An object of $\mathcal{G}$ is a pair (X, f) consisting of an object $X : \mathrm{T}_{\mathrm{pr}}$ and a morphism $f : S \to \rho(X)$ of sets.

Theorem 4. *There is a sound interpretation* $[\![-]\!]_{\mathcal{G}} : \mathrm{T}_{\mathrm{pr}} \to \mathcal{G}$*.*

As a corollary, all T_{pr}-definable functions between the natural numbers are primitive recursive. We use synthetic Tait computability to prove canonicity for T_{pr}, analogous to [4, § 4.5.3]. It's not exactly the same, because the type theory presented there does not contain a natural numbers object.

Future directions We then speculate on how to extend T_{pr} to a homotopy type theory. An obvious obstacle is that we can't just look at retracts of $\mathrm{y}_{\mathbb{N}}$, since these are all 0-truncated. One option is to try to develop the cubical model internally in T_{pr}.

References

[1] Hugo Herbelin and Ludovic Patey. 2014. A calculus of primitive recursive constructions. TYPES abstract. (2014). https://www.irif.fr/~letouzey/types2014/abstract-41.pdf.

[2] Martin Hofmann. 1997. An Application of Category-Theoretic Semantics to the Characterisation of Complexity Classes Using Higher-Order Function Algebras. *Bulletin of Symbolic Logic*, 3, 4, (Dec. 1997), 469–486. DOI: 10.2307/421100.

[3] Maria Emilia Maietti. 2010. Joyal's arithmetic universe as list-arithmetic pretopos. *Theory Appl. Categ.*, 24, No. 3, 39–83.

[4] Jonathan Sterling. 2021. *First Steps in Synthetic Tait Computability: The Objective Metatheory of Cubical Type Theory*. PhD Thesis. Carnegie Mellon University. DOI: 10.5281/zenodo.6990769. Issue: CMU-CS-21-142.