Tangent bundles and Euler classes

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We construct tangent bundles of spheres in Homotopy Type Theory and show the Hairy ball theorem: the tangent bundle admits a nowhere vanishing section if and only if the sphere is odd-dimensional.

We give both a direct proof of the Hairy ball theorem, as well as a proof via Euler classes, which we give a new presentation of.

Ideally, as in classical homotopy theory, we would like to have the tangent bundles of (the homotopy types of) n-manifolds represented by maps to $\mathrm{BO}(n)$, the classifying space of the orthogonal group $\mathrm{O}(n)$. But since it's still open to construct the types $\mathrm{BO}(n)$, we instead construct maps to $\mathrm{B}\mathcal{G}(n)$, the classifying type of the group of self homotopy equivalences of the $(n-1)$-sphere S^{n-1}. This can be defined simply as

$$\mathrm{B}\mathcal{G}(n) := \sum_{X:\mathcal{U}} \|X = S^{n-1}\|,$$

and a bundle $\xi : M \to \mathrm{B}\mathcal{G}(n)$ is then visibly a family of $(n-1)$-spheres. For any n-manifold M, the tangent bundle is then represented via the family of $(n-1)$-spheres of unit tangent vectors corresponding to the forgetful map $\mathrm{BO}(n) \to \mathrm{B}\mathcal{G}(n)$.

It is possible to define $\tau^n : S^n \to \mathrm{B}\mathcal{G}(n)$ simultaneously with a family of identifications $\theta^n : \prod_{x:S^n} \Sigma\tau^n(x) = S^n$ witnessing that the tangent bundle becomes trivial after adding a trivial line bundle. Since we are ultimately looking for a general construction of Grassmannians, which would help us define the types $\mathrm{BO}(n)$, we give a more general construction, abstracting the essential features of the situation, along the following lines:

Let A and E be types. We say that E is **a torsor for the type A** if E is inhabited and we have a map $T : E \to (A \simeq E)$. For example, any 2-element type is a torsor for the booleans/0-sphere S^0.

We say that A **has a reflection** if there exists $r : A \simeq A$ and $h : \mathrm{id}_A * r = \mathrm{sym}_{A,A}$, identifying two maps $A * A \to A * A$, where $*$ denotes the join operation and $\mathrm{sym}_{A,A}$ swaps the two join factors. Note that r is necessarily an involution, since $\mathrm{sym}_{A,A}$ is. In the case of S^0, we let r be the involution swapping the two points.

We say that A **has a coherent reflection** if in addition we have that $h(\mathrm{inl}\, a) = \mathrm{glue}(a, a) : \mathrm{inl}\, a = \mathrm{inr}\, a$ and $h(\mathrm{inr}\, a) = \mathrm{glue}(a, r(a))^{-1} : \mathrm{inr}(r(a)) = \mathrm{inl}\, a$ for each $a : A$. This is the case for the reflection on S^0.

Theorem 1. *Let A be a type with a reflection r and let E be a torsor for A with structure map T. Then for each $n : \mathbb{N}$ and $x : E^{*n}$, we have $\tau^n(x) : \mathcal{U}$ and an equivalence $\theta^n(x) : A * \tau^n(x) \simeq E^{*n}$. Furthermore, each $\tau^n(x)$ is merely equivalent to $E^{*(n-1)}$ and $A^{*(n-1)}$.*

Let E be a torsor for A with structure map T. For $a : A$, we get a map $\lambda e.T(e, a) : E \to E$. By functoriality of the join power, this gives a map $A \to E^{*n} \to E^{*n}$ which we call **scalar multiplication**. We write it as $a \cdot e$ for $a : A$ and $e : E^{*n}$. In the case of the spheres, considered as join powers of S^0, the scalar multiplication becomes the antipodal map (and the identity).

Theorem 2. *Let A be a type with a coherent reflection r and let E be a torsor for A with structure map T. Then we have a path*

$$H^n_{a,e} : \theta^n(e)(\mathrm{inl}\, a) = a \cdot e \text{ in } E^{*n}$$

for $n : \mathbb{N}$, $a : A$, and $e : E^{*n}$.

Theorem 3. *Let A be a type with a coherent reflection r and let E be a torsor for A with structure map T. Assume that there exists an element $1 : A$ such that $T(e, 1) = e$ for all $e : E$. Let $n : \mathbb{N}$ and assume that there exists a section $s : \prod_x \tau^n(x)$ of the tangent bundle of E^{*n}. Then $a \cdot x = x$ for all $a : A$ and $x : E^{*n}$.*

Corollary 4 (Hairy Ball Theorem)**.** *If n is even, then the tangent bundle of $(S^0)^{*(n+1)} \simeq S^n$ has no section.*

Proof. Assume that we have a section $s : \prod_x \tau^{n+1}(x)$, where x runs over $(S^0)^{*(n+1)}$. Note that in the torsor structure for S^0 over itself, we have that $T(e, 1) = e$ for all $e : S^0$. Thus, by Theorem 3, we have that $(-1) \cdot x = x$ for all $x : (S^0)^{*(n+1)}$. The map $(-1) \cdot -$ is by definition $r^{*(n+1)}$. Note that $r^{*2} = (\mathrm{id} * r) \circ (r * \mathrm{id}) = \mathrm{sym} \circ \mathrm{sym} = \mathrm{id}$. Therefore, since n is even, $r^{*(n+1)} = r * \mathrm{id}$. Under the equivalence of $S^0 * X$ with the suspension ΣX, $r * \mathrm{id}$ corresponds to the self-equivalence that reverses N and S in the suspension, which gives negation in homotopy groups. So $r * \mathrm{id}$ corresponds to $(-1) : \pi_{n+1}(S^{n+1}) \simeq \mathbb{Z}$. This contradicts the fact that $r * \mathrm{id}$ is homotopic to the identity map. □

We have formalized these results using the Coq-HoTT library.

Euler classes We also develop an alternative approach to the Hairy ball theorem via Euler classes. By proving that the Euler class of τ^n is 2 in even dimensions and 0 in odd dimensions, the Hairy ball theorem follows. Euler classes are defined for *oriented* sphere bundles.

We build on [BCFR23] to define orientations. If A is S^n or $\mathrm{K}(\mathbb{Z}, n)$ and X is any type identifiable with A, then $\|A = X\|_0$ is a 2-element set, the **the set of orientations of X**. Let $\mathrm{BAut}_+(X) :\equiv \Sigma_{X:\mathrm{BAut}(A)} \|A = X\|_0$ be the type of oriented types identifiable with A. By [BCFR23, Prop. 5.9], for $n > 1$, we can identify the type $\mathrm{BAut}_+(\mathrm{K}(\mathbb{Z}, n-1))$ with $\mathrm{K}(\mathbb{Z}, n)$. This then yields a particularly direct way of defining Euler classes: The **(universal) Euler class** is the map

$$\mathrm{e}^{\mathbb{Z}} : \mathrm{BAut}_+(S^{n-1}) \to \mathrm{BAut}_+(\mathrm{K}(\mathbb{Z}, n-1))$$

induced by $(n-1)$-truncation.

Theorem 5. *Let $n > 1$ and let $P : X \to \mathrm{BAut}_+(S^{n-1})$ be an oriented sphere bundle on a type X. If P merely has a section, then $\mathrm{e}^{\mathbb{Z}}(P) = 0$.*

We also prove that this definition of the Euler class is correct since it agrees with the pullback of the Thom class along the zero section map from the base to the Thom space.

Time permitting, we will also discuss twisted Euler classes and further results related to our work towards defining homotopy manifolds in HoTT.

References

[BCFR23] U. Buchholtz, J. D. Christensen, J. G. T. Flaten, and E. Rijke. *Central H-spaces and banded types.* 2023. arXiv: 2301.02636v1.