Towards computable homotopy theory

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Already in [3] Hyland showed an interesting connection between topos theory and computability theory: the Turing degrees embed into the lattice of subtoposes of the effective topos. We can update this idea to work in HoTT through modalities [4] and cubical assemblies [6, 5]. We can think of functions $\chi : \mathbb{N} \to \nabla\mathbb{N}$ where ∇ is the lex double negation modality, as a way to talk about non computable functions in a setting where all functions $\mathbb{N} \to \mathbb{N}$ are computable. We define the *oracle modality* $\bigcirc_\chi$ to be the smallest modality that forces χ to be a total function $\mathbb{N} \to \mathbb{N}$. We can think of functions $\mathbb{N} \to \bigcirc_\chi\mathbb{N}$ as functions (possibly non computable) that can be computed using an oracle Turing machine with oracle χ. This can be made precise using cubical assemblies, where every function $\mathbb{N} \to \mathbb{N}$ in sets appears as a function $\mathbb{N} \to \nabla\mathbb{N}$ in cubical assemblies, and two functions χ, χ' have the same Turing degree if and only if there is a closed term witnessing that the modalities $\bigcirc_\chi$ and $\bigcirc_{\chi'}$ are equal in cubical assemblies.

In this talk I will focus on using oracle modalities to provide some promising connections between computability theory and homotopy theory. The first connection is a proof that two Turing degrees are equal if and only if the groups of permutations on $\mathbb{N}$ computable in each degree are isomorphic. Although this can also be proved directly, we can use HoTT to give a new proof using ideas from homotopy theory. In particular, we make use of the elegant formulation of group theory and in particular wreath product in HoTT [1].

The second connection is some work in progress exploring the *suspension* of oracle modalities. Given any modality $\bigcirc$, one can define another modality $\bigcirc^=$ such that a type is $\bigcirc^=$-modal iff it is $\bigcirc$-separated [2]. We can think of $\bigcirc_\chi X$ as the type consisting of elements of X where we are allowed to use the oracle χ to construct them. In other words we can use the modality χ to construct new points of X that would not otherwise be computable. On the other hand in $\bigcirc_\chi^= X$ we can use the oracle χ to compute new paths in X, without adding any new points. I will talk about some preliminary work looking at the effect of the suspensions of oracle modalites on homotopy groups of higher types.

References

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