Univalent Double Categories

Nima Rasekh joint with Niels van der Weide, Benedikt Ahrens, Paige Randall North

Formalization of mathematics is often seen as a way to formally verify mathematical definitions and theorems and gain more confidence in their veracity. However, in certain mathematical contexts and appropriately chosen type theories, pursuing formalization can result in additional advantages that can broaden our mathematical perspective. In this talk we will focus on the particular case of the **Univalence Maxim** and its application to the study and formalization of **double category theory**.

1 The Univalence Maxim for Categorical Structures

The *univalence maxim* broadens our understanding of categorical structures via formalization in Coq UniMath, by relating categorical structures and various notions of equivalences. Before providing a more precise characterization, we will first examine two relevant examples.

Categories were first formalized in Coq UniMath by Ahrens–Kapulkin–Shulman [AKS15]. As part of their formalization they also defined *univalent* categories and established a *univalence principle*, proving that identities of the type of categories coincides with equivalences. Moreover, as an immediate implication of their argument it follows that the type of categories with a set of objects has identities given by isomorphisms of categories. We hence witness that in Coq UniMath we have *two* notions of categories: categories with a set of objects, who are invariant under isomorphisms, and univalent categories, who are invariant under equivalences. We are hence witnessing a correspondence between possible formalizations of categories in UniMath and possible notions of equivalences of categories. This should be understood as a first manifestation of the above-mentioned maxim.

The formalization of categories has been generalized by Ahrens and several other authors to a formalization of 2-categories and bicategories in Coq UniMath [AFM+21]. The authors in particular formalize univalent bicategories and establish their univalence principle, by proving that identities in the type of univalent bicategories coincide with biequivalences of bicategories. One implication of this result is that we need to relax the categorical structure from a 2-category, where the composition of 1-morphisms is strictly associative and unital, to a bicategory, meaning weaken those unitality and associativity conditions, in order to prove the univalence principle. Using similar methods, we can prove a univalence principle for 2-categories with a set of objects and univalent hom categories, by observing that their identities coincide with functors that are isomorphisms of objects and local equivalences of hom categories.

Here for the first time we are witnessing that obtaining a univalence principle for a given choice of equivalence of a categorical structure, can necessitate adjusting, and particularly weakening, the categorical structure. The examples we have presented allow us to now articulate the **Univalence Maxim for Categorical Structures** in a more precise manner: For every categorical structure and for every possible notion of equivalence of that structure, there exists a corresponding notion of univalent categorical structure, such that its identities precisely correspond to the chosen equivalences. We can in particular understand this maxim as a feedback loop between category theory and univalent mathematics. Indeed, advancing our knowledge of categories benefits from formalizing it (an application of formalization to category theory), and formalizing categories along with a chosen equivalence in Coq UniMath requires understanding possible weakening of the chosen categorical structure (an application of categorical literature to formalization).

2 The Maxim in Action: Double Categories

In a variety of situations objects witness more than one relevant notion of morphism. Important examples include sets, which come with functions and relations, categories, with functors and profunctors, or rings, with ring homomorphisms and modules. This motivates defining a categorical structure which generalizes categories and can capture the data of two types of morphisms, which is known as a *double category*.

Double categories are a categorical structure consisting of objects and two types of morphisms (called horizontal and vertical morphisms) that interact well with each other via appropriately chosen squares [Ehr63]. They were introduced by Ehresmann as a tool to better understand categories, and played an important role in formal category theory and the theory of equipments [Woo82, Woo85]. Beyond those original applications, double categories have also found a variety of applications in applied mathematics and computer science; see, for instance, its applications in systems theory [Cou20, Mye21, BCV22] and programming languages theory [DM13, NL23]. As part of this project we discuss the formalization of double categories, their results, and their examples, as well as provide further evidence for the *univalence maxim* in the context of the formalization of double categorical notions in Coq UniMath.

As the definition of a double category involves far more data than a category, double categories exhibit many different notions of equivalences. This includes standard notions such as an isomorphism of double categories. However, we can also define a *horizontal equivalence* defined as inducing equivalences on the following two underlying categories: the one given by objects and horizontal morphisms, and the one given by vertical morphisms and squares. Similarly, we can define *vertical equivalences*. Moreover, we can generalize both notions to *horizontal (vertical) biequivalences* [MSV20]. Finally we also have symmetric notions of equivalences, such as *gregarious equivalences* [Cam20, ANST21].

As part of our work, we apply the univalence maxim to double categorical structures. Using this method we obtain a correspondence between invariances of double categories and corresponding double categorical structures with appropriately chosen strictness of unitality and associativity, which we also formalize in Coq UniMath. We can summarize it as follows:

Invariance	**Structure**	**Source**	**Formalization**
Iso. of Double Cat.	Set Double Cat.	[Ehr63]	[Weic]
Iso. of Pseudo Double Cat.	Set Pseudo Cat.	[Gra20]	[Weia]
Horizontal Equiv. of Double Cat.	Univalent Pseudo Double Cat.	[Gra20]	[vdWRAN24, Weib]
Gregarious Equiv. of Double Cat.	Univalent Double Bicat.	[Ver11, ANST21]	[RWAN24, Weid]

Here, a pseudo-double category has strict horizontal composition but only weakly associative and unital vertical composition. It is univalent if the two categories given by objects and horizontal morphisms and vertical morphisms and squares are univalent [vdWRAN24]. Moreover, double bicategories have weakly associative and unital compositions in both directions, fitting the symmetric nature of gregarious equivalences. Its categorical properties and formalization are part of the follow up paper [RWAN24], building on the univalence principle developed in [ANST21].

3 Conclusion

Our work has two major accomplishments:

1. **Application to Category Theory:** Applying the univalence maxim motivated by Coq UniMath to obtain a structured understanding of double categorical equivalences.

2. **Application to Formalization:** Formalizing a wide range of double categorical notions, their properties and their univalence principles.

Our work suggests further possible applications to both the classification as well as formalization of other categorical structures, such as 3-categories or symmetric monoidal categories.

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