Two-sided Fibration Categories

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Background

Directed type theory seeks to expand the correspondence between types and ∞-groupoids, in order to include non-groupoidal higher categories. Various syntactic approaches have been proposed [5, 6, 7]. Riehl and Shulman [9] have proposed a type theory that allows "probing with a directed interval type". This approach has been further developed by Buchholtz and Weinberger [4], and Weaver and Licata [10] proposed a cubical variant. Alternkirch and Neumann [2, 1] propose a semantic approach based on polarized category with families; these build on categories with families by requiring a context negation operation modeled by taking a category A to its opposite.

In this talk, we will propose a novel semantic framework based on abstract classes of *directed fibrations*. Our hope is that the study of these semantic objects will inform the syntax of future directed type theories.

Directed Path Objects

A fundamental insight of homotopy type theory is that categories with a distinguished notion of fibration can be used to model Martin-Löf dependent type theory [3]. In particular, identity types can be interpreted as path objects. Thus, in order to understand the semantics of *homomorphism types* in a directed type theory, we consider *directed path object factorisations*.

$$A \longrightarrow \mathrm{Hom}_A \twoheadrightarrow A \times A.$$

We will introduce a notion of *two-sided fibration category*, which consists of a well-behaved category of fibrant objects, together with distinguished classes of covariant and contravariant fibrations. We additionally require directed path object factorisations, and lifting properties closely connected to those described by van den Berg, North and McCloskey [8].

Our notion has a number of appealing properties. Firstly, every model of undirected type theory induces an undirected two-sided fibration category. Secondly, every suitably structured 2-category induces a two-sided fibration category. For example, in the category of categories, contravariant fibrations are Grothendieck fibrations, and covariant fibrations as opfibrations. In this interpretation, the directed path object Hom_A of a category A is its arrow category $\vec{A}$. Finally, every object in a two-sided fibration category can be equipped with the structure of a weak ω-category.

Challenges

Our talk will highlight some of the complications that arise compared to undirected fibration categories. These difficulties can be already be seen by considering the category of categories.

A first obstacle is that while the category of isofibrations in $\mathscr{C}$ over A is a full subcategory of $\mathscr{C}/A$, categories of directed fibrations are usually not full. For example, we typically require morphisms between Grothendieck fibrations to be *cartesian*. Consequently, our definition allows non-full subcategories of fibrations, and requires axioms to ensure that enough arrows are morphisms of fibrations.

A second challenge is that iteratively taking fibrations of different variance is non-trivial. If $\mathscr{C}$ is an (undirected) category of fibrant objects, then each fibrational slice $\mathrm{Fib}_A(\mathscr{C})$ is also a category of fibrant objects. Yet, in our framework the category $\mathrm{Fib}^+_A(\mathscr{C})$ of covariant fibrations does not automatically inherit a sensible notion of covariant fibration. We certainly do not expect that $\mathrm{Fib}^-_B(\mathrm{Fib}^+_A(\mathscr{C})) = \mathrm{Fib}^-_B(\mathscr{C})$. We address this by coinductively requiring the structure of a two-sided fibration category on $\mathrm{Fib}^+_A(\mathscr{C})$ and $\mathrm{Fib}^-_A(\mathscr{C})$. We then require *compatibility axioms* that help us to tame the resulting explosion of data.

Finally, the contexts that we would like to consider in a directed type theory can depend covariantly and contravariantly on the same variable, and this complicates the rules for homomorphism types. Let us describe a more abstract way to understand this difficulty. To start with, we expect homomorphism types to be the units of the composition of profunctors; this is the universal property of the Hom profunctor, and it is effectively the statement that natural transformations out of Hom_A are determined by the identity arrows of A. Thus, our theory needs to be able to reason about composition of profunctors. However, while two-sided fibrations correspond to profunctors $A^{\mathrm{op}} \times B \to \mathrm{Set}$, composition of spans is not composition of profunctors. We tackle this issue by introducing a notion of *equivariant morphisms*; these allow us to reason about extranaturality conditions without computing coends.

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