A topological reading of (co)inductive definitions in Dependent Type Theories

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In this work, we extend a correspondence previously proved in [7] between W-types and proof-relevant inductive basic covers by establishing a similar correspondence between proof-relevant coinductive positivity relations and a particular subclass of M-types.

Basic covers and positivity relations constitute the main components of a point-free notion of topological space called *basic topology* by primitively representing its open and closed subsets, respectively. Basic topologies are the fundamental object investigated by Formal Topology, which is the study of topology in a constructive and predicative setting. Powerful techniques for inductively generating basic covers and coinductively generating positivity relations have been developed in [4, 12] and have since been a cornerstone of the field.

Our work compares these topological (co)inductive methods with other established type-theoretic schemes for (co)induction through an intermediate notion, that of *inductive and coinductive predicates*, born in the context of axiomatic set theories [1, 11] and adapted by us to various dependent type theories. Firstly, we defined them as propositions in the Minimalist Foundation [8, 6], a foundational theory designed as a common core among various foundational theories for constructive mathematics that primitively assumes a notion of proposition. The Minimalist Foundation can then be interpreted, on the one hand, into intensional Martin-Löf's type theory [9] by enforcing the proposition-as-type paradigm; on the other, in Homotopy Type Theory [10], interpreting propositions as h-propositions [3]. Thus, in Homotopy Type Theory, we obtain two versions of (co)inductive predicates: a proof-relevant version, as generic types coming from the interpretation in Martin-Löf's type theory, and a proof-irrelevant version, as h-propositions.

Using function extensionality, we prove that, in Homotopy Type Theory, proof-relevant inductive basic covers are equivalent to W-types and proof-relevant coinductive positivity relations are particular M-types. Thus, they are both supported in Homotopy Type Theory by the results in [2]. Similar correspondences can also be proved for their proof-irrelevant versions without extensionality principles. However, while it is known that proof-irrelevant inductive basic covers are supported by Homotopy Type Theory using Higher Inductive Types [5], to the best of our knowledge, it is an open problem to give an internal description of coinductive h-propositions.

References

[1] Peter Aczel. "An Introduction to Inductive Definitions". In: *HANDBOOK OF MATHEMATICAL LOGIC*. Ed. by Jon Barwise. Vol. 90. Studies in Logic and the Foundations of Mathematics. Elsevier, 1977, pp. 739–782. DOI: https://doi.org/10.1016/S0049-237X(08)71120-0. URL: https://www.sciencedirect.com/science/article/pii/S0049237X08711200.

[2] Benedikt Ahrens, Paolo Capriotti, and Régis Spadotti. "Non-wellfounded trees in homotopy type theory". In: *13th International Conference on Typed Lambda Calculi and Applications*. Vol. 38. LIPIcs. Leibniz Int. Proc. Inform. Schloss Dagstuhl. Leibniz-Zent. Inform., Wadern, 2015, pp. 17–30. ISBN: 978-3-939897-87-3.

[3] Michele Contente and Maria Emilia Maietti. "The Compatibility of the Minimalist Foundation with Homotopy Type Theory". In: 2022. URL: https://api.semanticscholar.org/CorpusID:250408058.

[4] T. Coquand et al. "Inductively generated formal topologies." In: *Annals of Pure and Applied Logic* 124.1-3 (2003), pp. 71–106.

[5] Thierry Coquand and Ayberk Tosun. "Formal Topology and Univalent Foundations". In: *Proof and Computation II*. Chap. Chapter 6, pp. 255–266. DOI: 10.1142/9789811236488_0006. eprint: https://www.worldscientific.com/doi/pdf/10.1142/9789811236488_0006. URL: https://www.worldscientific.com/doi/abs/10.1142/9789811236488_0006.

[6] Maria Emilia Maietti. "A minimalist two-level foundation for constructive mathematics". In: *Ann. Pure Appl. Logic* 160.3 (2009), pp. 319–354. ISSN: 0168-0072,1873-2461. DOI: 10.1016/j.apal.2009.01.006. URL: https://doi.org/10.1016/j.apal.2009.01.006.

[7] Maria Emilia Maietti and Pietro Sabelli. "A topological counterpart of well-founded trees in dependent type theory". In: *Electronic Notes in Theoretical Informatics and Computer Science* Volume 3 - Proceedings of MFPS XXXIX (Nov. 2023). DOI: 10.46298/entics.11755. URL: https://entics.episciences.org/11755.

[8] Maria Emilia Maietti and Giovanni Sambin. "Toward a minimalist foundation for constructive mathematics". In: *From sets and types to topology and analysis*. Vol. 48. Oxford Logic Guides. Oxford Univ. Press, Oxford, 2005, pp. 91–114. DOI: 10.1093/acprof:oso/9780198566519.003.0006. URL: https://doi.org/10.1093/acprof:oso/9780198566519.003.0006.

[9] B. Nordström, K. Petersson, and J. Smith. *Programming in Martin Löf's Type Theory.* Clarendon Press, Oxford, 1990.

[10] The Univalent Foundations Program. *Homotopy type theory—univalent foundations of mathematics*. The Univalent Foundations Program, Princeton, NJ; Institute for Advanced Study (IAS), Princeton, NJ, 2013, pp. xiv+589.

[11] Michael Rathjen. "Generalized inductive definitions in constructive set theory". In: *From sets and types to topology and analysis*. Vol. 48. Oxford Logic Guides. Oxford Univ. Press, Oxford, 2005, pp. 23–40. DOI: 10.1093/acprof:oso/9780198566519.003.0002. URL: https://doi.org/10.1093/acprof:oso/9780198566519.003.0002.

[12] Giovanni Sambin. "Dynamics in Foundations: What Does It Mean in the Practice of Mathematics?" In: *Reflections on the Foundations of Mathematics: Univalent Foundations, Set Theory and General Thoughts*. Ed. by Stefania Centrone, Deborah Kant, and Deniz Sarikaya. Cham: Springer International Publishing, 2019, pp. 455–494. ISBN: 978-3-030-15655-8. DOI: 10.1007/978-3-030-15655-8_21. URL: https://doi.org/10.1007/978-3-030-15655-8_21.