RECENT PROGRESS IN THE THEORY OF EFFECTIVE KAN FIBRATIONS IN SIMPLICIAL SETS

BENNO VAN DEN BERG

Constructivity issues in simplicial sets. One of the most important steps in the development of homotopy type theory has been the construction, by Voevodsky, of a model of type theory with the univalence axiom in the category of simplicial sets [10]. This work builds on the classic Kan-Quillen model structure in simplicial sets.

From the very beginning people have been trying to understand how *constructive* Voevodsky's results are. Besides being of intrinsic interest, it is also relevant for the question whether these results hold relative to an arbitrary base topos. Perhaps most importantly it also asks whether one can *compute* with the univalence axiom, or any other principle that might hold in the simplicial model.

Early on, an obstruction was found by Bezem, Coquand and Parmann [4]. They observed that the classic result saying that the exponential A^B is a Kan complex whenever A and B are, is not provable constructively. We refer to this as the *BCP-obstruction.* Since Kan complexes are interpreting the types in Voevodsky's model and the exponentials are the obvious way to interpret function spaces, this blocks a direct constructive interpretation of function types in Voevodsky's model.

Indeed, the best results that we have in this direction can be found in the work of Gambino, Henry, Sattler and Szumilo [9, 6, 8]. After Henry showed that the Kan-Quillen model structure can be proven to exist constructively, these authors showed that there are basically two obstacles to obtaining a constructive account of Voevodsky's model. First, one would only have weak function types, with rules weaker than the usual ones, due to the BCP-obstruction. Secondly, to obtain a genuine of model theory, a difficult coherence problem needs to be solved for which currently no solution is known.

Cubical type theory. In response most researchers have switched to *cubical sets.* This does not only involve changing the shapes, but also involves strengthening the notion of a Kan comples, or a Kan fibration, by adding *uniformity conditions* [3, 5].

Indeed, the usual definition of a Kan complex requires the mere existence of fillers against a class of maps, whether these are horn inclusions or open box inclusions.

Institute for Logic, Language and Computation, Universiteit van Amsterdam, Postbus 90242, 1090 GE Amsterdam, The Netherlands

E-mail address: B.vandenBerg3@uva.nl.

Date: February 25, 2024.

The other innovation is to insist that a Kan fibration comes equipped with a system of solutions which is required to satisfy certain compatibility conditions. It is in this way that one can overcome the BCP-obstruction in cubical sets.

Back to simplicial sets. While this has sometimes been taken to mean that cubical sets are constructively superior, matters are really not that clear. Indeed, as observed by Gambino and Sattler [7], uniformity conditions can also be used to overcome the BCP-obstruction in simplicial sets. Indeed, in their paper they define a notion of a uniform Kan complex, mirroring the cubical definition, and show that if A is a uniform Kan complex, then so is A^B for any simplicial set B.

Effective Kan fibrations. In a book written with Eric Faber [1], we gave another solution which we call *effective Kan fibrations*, using uniformity conditions stronger than those of Gambino and Sattler. In contrast to Gambino and Sattler's notion, our definition is *local*. This means that we can show the existence of universal effective Kan fibrations, which should allow us to interpret type-theoretic universes. Indeed, the main results of our book are:

(1) Every effective Kan fibration is a Kan fibration in the usual sense, and in a classical metatheory one can show that every Kan fibration can be equipped with the structure of an effective Kan fibration.
(2) Whenever f and g are effective Kan fibrations, then so is $\Pi_f(g)$, the push forward of g along f.
(3) Being an effective Kan fibration is a local property and hence universal effective Kan fibrations exist.

Recent progress. The ultimate aim is to develop a constructive proof of the existence of both a model of homotopy type theory and the Kan-Quillen model structure on simplicial sets using the notion of an effective Kan fibrations. Unfortunately, this remains work in progress.

In the meantime, the speaker has obtained, often together with (former) MSc students, some further results and the purpose of this talk is to report on these.

In particular, we have shown that:

(1) Any simplicial group is effectively Kan ([2], jww with Freek Geerligs).
(2) The effective Kan fibrations are cofibrantly generated by a countable double category ([2], jww with Freek Geerligs). Classically, this means they are the right class in algebraic weak factorisation system.
(3) Whenever f is an effective Kan fibration, then W_f, the W-type associated to f is an effective Kan complex (jww with Shinichiro Tanaka).
(4) A version of the Joyal-Tierney calculus works for effective Kan fibrations (jww with Eric Faber).

Since we are still working on related issues, we may have more to report in April.

References

[1] Benno van den Berg and Eric Faber. *Effective Kan fibrations in simplicial sets*. Vol. 2321. Lecture Notes in Mathematics. Springer, Cham, [2022] ©2022, pp. x+230.

[2] Benno van den Berg and Freek Geerligs. "Examples and cofibrant generation of effective Kan fibrations". arXiv:2402.10568. 2024.

[3] Marc Bezem, Thierry Coquand, and Simon Huber. "The Univalence Axiom in Cubical Sets". *J. Autom. Reasoning* 63.2 (2019), pp. 159–171.

[4] Marc Bezem, Thierry Coquand, and Erik Parmann. "Non-constructivity in Kan simplicial sets". *13th International Conference on Typed Lambda Calculi and Applications*. Vol. 38. LIPIcs. Leibniz Int. Proc. Inform. Schloss Dagstuhl. Leibniz-Zent. Inform., Wadern, 2015, pp. 92–106.

[5] Cyril Cohen et al. "Cubical Type Theory: A Constructive Interpretation of the Univalence Axiom". *FLAP* 4.10 (2017), pp. 3127–3170.

[6] Nicola Gambino and Simon Henry. "Towards a constructive simplicial model of univalent foundations". Accepted for publication in *Journal of the London Mathematical Society*. arXiv:1905.06281. 2019.

[7] Nicola Gambino and Christian Sattler. "The Frobenius condition, right properness, and uniform fibrations". *J. Pure Appl. Algebra* 221.12 (2017), pp. 3027–3068.

[8] Nicola Gambino, Christian Sattler, and Karol Szumiło. "The constructive Kan-Quillen model structure: two new proofs". Accepted for publication in *The Quarterly Journal of Mathematics*.arXiv:1907.05394. 2019.

[9] Simon Henry. "A constructive account of the Kan-Quillen model structure and of Kan's Ex^∞ functor". arXiv:1905.06160. 2019.

[10] Krzysztof Kapulkin and Peter LeFanu Lumsdaine. "The simplicial model of univalent foundations (after Voevodsky)". *J. Eur. Math. Soc. (JEMS)* 23.6 (2021), pp. 2071–2126.