Cellular Homology and the Cellular Approximation Theorem

Axel Ljungström
Stockholm University, Sweden
axel.ljungstrom@math.su.se

Anders Mörtberg
Stockholm University, Sweden
anders.mortberg@math.su.se

Loïc Pujet
Stockholm University, Sweden
loic@pujet.fr

In [BH18], Buchholtz and Favonia develop a theory of cellular cohomology in HoTT. The authors proceed in two steps: first, they define the cohomology groups of a CW complex, employing the standard definition in terms of cochain complexes (see e.g. [May99]), and then they construct an isomorphism between their definition and the, in HoTT, more well-established cohomology theory defined in terms of Eilenberg-MacLane spaces [Shu13; LF14; Cav15]. This second step allows the authors to derive many properties of their cohomology theory (e.g. functoriality and the Eilenberg-Steenrod axioms) simply by transporting the relevant proofs from Eilenberg-MacLane cohomology. However, this strategy is not as readily available when developing cellular *homology*: even though we can define homology theories in terms of Eilenberg-MacLane spaces in HoTT [Gra18; CS23; Doo18; Spe18], this is significantly more involved than for cohomology as it relies on the theory of stable homotopy groups and smash product spectra. This suggests that, perhaps, a direct construction of cellular homology is the more feasible alternative.

In this work, we revisit Buchholtz and Favonia's definition of cellular chain complexes from which we define a functorial homology theory. This is done not via reduction to another more well-studied definition, but by developing the theory of CW complexes and cellular maps. In particular, we prove a constructive version of the *cellular approximation theorem*, a cornerstone of the classical theory of CW complexes. All results presented here have been formalised in `Cubical Agda` [VMA21].

We will need the following definition to define CW complexes:[1]

Definition 1. *A* ***CW-skeleton*** *is an infinite sequence of types and maps* $(X_{-1} \xrightarrow{\mathsf{incl}_{-1}} X_0 \xrightarrow{\mathsf{incl}_0} X_1 \xrightarrow{\mathsf{incl}_1} \dots)$ *equipped with a function* $c : \mathbb{N} \to \mathbb{N}$ *and a set of attaching maps* $\alpha_i : S^i \times \mathsf{Fin}(c_{i+1}) \to X_i$ *for* $i \geq -1$ *s.t.* X_{-1} *is empty and the square on the right is a (homotopy) pushout. A CW-skeleton is said to be* ***finite (of dimension*** n***)*** *if* incl_m *is an equivalence for all* $m \geq n$.

$$\begin{array}{ccc} S^i \times \mathsf{Fin}(c_{i+1}) & \xrightarrow{\mathsf{snd}} & \mathsf{Fin}(c_{i+1}) \\ {\scriptstyle \alpha_i}\downarrow & & \downarrow \\ X_i & \xrightarrow{\qquad\ulcorner} & X_{i+1} \end{array}$$

The pushout condition ensures that the $(i+1)$-skeleton X_{i+1} is obtained by attaching a finite number of i-dimensional cells to the i-skeleton X_i. We will often simply write $X_\bullet$ for a CW-skeleton $(X_0, X_1, \dots)$ and take $\mathsf{incl}_\bullet$, $c_\bullet$ and $\alpha_\bullet$ to be implicit. We denote by $\mathsf{CW}^{\mathsf{skel}}_\infty$ the wild category whose objects are CW-skeleta and whose morphisms are maps between their sequential colimits, i.e. $\mathsf{Hom}(X_\bullet, Y_\bullet) := (X_\infty \to Y_\infty)$. We denote by $\mathsf{CW}^{\mathsf{skel}}$ the wild category with the same objects but whose morphisms are *cellular maps*:

Definition 2. *Let* $X_\bullet$ *and* $Y_\bullet$ *be CW-skeleta. A* ***cellular map****, denoted* $f_\bullet : X_\bullet \to Y_\bullet$*, consists of a family* $f_i : X_i \to Y_i$ *for* $i \geq -1$ *along with a family of homotopies* h_i *making the diagram on the right commute.*

$$\begin{array}{ccc} X_{i+1} & \xrightarrow{f_{i+1}} & Y_{i+1} \\ \uparrow & \overset{h_i}{\Longleftarrow} & \uparrow \\ X_i & \xrightarrow[f_i]{} & Y_i \end{array}$$

Definition 3. *A type* A *is said to be a* ***CW complex*** *if there merely exists some CW-skeleton* $X_\bullet$ *s.t.* A *is equivalent to the sequential colimit of* $X_\bullet$*, i.e.* $A \simeq X_\infty$.

Let $\mathbb{Z}[n]$ denote the free abelian group with $n-1$ generators, with $\mathbb{Z}[0]$ defined to be the trivial group. Buchholtz and Favonia [BH18] showed how to construct the chain complex associated to a CW-skeleton: $\dots \xrightarrow{\partial_3} \mathbb{Z}[c_2] \xrightarrow{\partial_2} \mathbb{Z}[c_1] \xrightarrow{\partial_1} \mathbb{Z}[c_0] \xrightarrow{\partial_0} 0$. We can show that $\partial_n \circ \partial_{n+1} = 0$ for all n, which allows us to define the n-th homology group of a CW-skeleton by $H^{\mathsf{skel}}_n(X) := \mathsf{ker}(\partial_n)/\mathsf{im}(\partial_{n+1})$. The differentials ∂_n are defined in terms of (an appropriate definition of) the *degree* of maps between

[1] This definition is slightly different from the recursive definition employed in (the formalisation of) [BH18]. Its usefulness is two-fold: first, it allows us to also define infinite dimensional CW complexes, such as $\mathbb{R}P^\infty$. Second, it allows us to extract the n-skeleton, X_n, of $X_\bullet$ directly without having to rely on auxilliary functions. A similar reformulation can be found in `https://github.com/CMU-HoTT/serre-finiteness/blob/cellular/Cellular/CellComplex.agda`.

wedge sums of spheres $\bigvee_{x:\mathsf{Fin}(c_{n+1})} \mathbb{S}^{n+1} \to \bigvee_{x:\mathsf{Fin}(c_n)} \mathbb{S}^{n+1}$. In fact, much of our work can be reduced to statements about the behaviour of such functions and of the degree assignment.

The homology groups defined here can be extended to functors from $\mathsf{CW}^{\mathsf{skel}}$ to AbGrp:[2]

Proposition 1. *H_n^{skel} is functorial.*

The argument is standard. We can transform a cellular map into an intertwining map between chain complexes, from which we get a homomorphism of homology groups.

Now, in order to get a homology theory on CW complexes, we would like to lift this homology functor from $\mathsf{CW}^{\mathsf{skel}}$ to $\mathsf{CW}^{\mathsf{skel}}_\infty$. We can straightforwardly define $H_n^{\mathsf{skel}_\infty} : \mathsf{CW}^{\mathsf{skel}}_\infty \to \mathsf{AbGrp}$ on objects by $H_n^{\mathsf{skel}_\infty}(X) := H_n^{\mathsf{skel}}(X)$. The action on morphisms, however, is less obvious: in order to reuse the functoriality of H_n^{skel}, we need a way to lift maps $X_\infty \to Y_\infty$ to cellular maps $X_\bullet \to Y_\bullet$. In the classical theory of CW complexes, this is the role of the *cellular approximation theorem* [May99, Section 10.4]. However, this theorem critically relies on the axiom of choice, so we cannot prove it as is if we want to be constructive. Fortunately, we are still allowed to use finite choice, which lets us prove a version which is restricted to the case where $X_\bullet$ and $Y_\bullet$ are finite:

Theorem 1 (Cellular approximation, part 1)**.** *Let $X_\bullet$ and $Y_\bullet$ be finite CW-skeleta and $f : X_\infty \to Y_\infty$ a map between their colimits. There merely exists a cellular map $f_\bullet : X_\bullet \to Y_\bullet$ s.t. $f_\infty = f$.*

By Theorem 1, it suffices to define the functorial action of $H_n^{\mathsf{skel}_\infty}$ on functions (of finite complexes) $f : X_\infty \to Y_\infty$ s.t. f is merely equal to f_∞ for some cellular map $f_i : X_\bullet \to Y_\bullet$. By the rule of set-valued elimination of propositional truncations [Kra15, Proposition 2], it suffices to define $H_n^{\mathsf{skel}_\infty}(f_\infty)$ for $f_i : X_\bullet \to Y_\bullet$ and prove that, if $f_\infty = g_\infty$, then $H_n^{\mathsf{skel}_\infty}(f_\infty) = H_n^{\mathsf{skel}_\infty}(g_\infty)$. We define $H_n^{\mathsf{skel}_\infty}(f_\infty) := H_n^{\mathsf{skel}}(f_\bullet)$. To complete the definition, we need to show that, if $f_\infty = g_\infty$, then $H_n^{\mathsf{skel}}(f_\bullet) = H_n^{\mathsf{skel}}(g_\bullet)$. Thus, we need to extend the approximation theorem to *cellular homotopies*:

Definition 4. *A **cellular homotopy** between cellular maps $f_\bullet, g_\bullet : X_\bullet \to Y_\bullet$, denoted $f_\bullet \sim g_\bullet$, is a family $h_i : (x : X_i) \to \mathsf{incl}_i(f_i(x)) =_{Y_{i+1}} \mathsf{incl}_i(g_i(x))$ with fillers, for each $x : X_i$, of the square on the right.*

$$\begin{array}{ccc} \mathsf{incl}_{i+1}(f_{i+1}(\mathsf{incl}_i(x))) & \xrightarrow{h_{i+1}(\mathsf{incl}_i(x))} & \mathsf{incl}_{i+1}(f_{i+1}(\mathsf{incl}_i(x))) \\ \uparrow & & \uparrow \\ \mathsf{incl}_{i+1}(\mathsf{incl}_i(f_i(x))) & \xrightarrow[\mathsf{ap}_{\mathsf{incl}}(h_i(x))]{} & \mathsf{incl}_{i+1}(\mathsf{incl}_i(g_i(x))) \end{array}$$

Proposition 2. *If $\| f_\bullet \sim g_\bullet \|$, then $H_n^{\mathsf{skel}}(f_\bullet) = H_n^{\mathsf{skel}}(g_\bullet)$.*

This follows from a technical, but standard, argument. The final component is:

Theorem 2 (Cellular approximation, part 2)**.** *Let $X_\bullet$ and $Y_\bullet$ be finite CW-skeleta with two cellular maps $f_\bullet, g_\bullet : X_\bullet \to Y_\bullet$ s.t. $f_\infty = g_\infty$. In this case, there merely exists a cellular homotopy $f_\bullet \sim g_\bullet$.*

Combining Theorem 2 and Proposition 2, we see that if $f_\infty = g_\infty$, then $H_n^{\mathsf{skel}}(f_\bullet) = H_n^{\mathsf{skel}}(g_\bullet)$, which completes the definition of the functorial action of $H_n^{\mathsf{skel}_\infty}$ on maps between finite complexes. In order to extend this to maps between possibly infinite complexes, one simply notes that $H_n^{\mathsf{skel}_\infty}(X_\bullet) \cong H_n^{\mathsf{skel}_\infty}(X_\bullet^{(n+2)})$ where $X_\bullet^{(m)}$ denotes the finite subcomplex of X_i, converging at level m.

Thus, we have defined the functor $H_n^{\mathsf{skel}_\infty}$, assigning homology groups to any type equivalent to the colimit of a CW-skeleton. However, for CW complexes, the existence of such an equivalence is only assumed to *merely* exist. We would like to define a fuctor $H_n^{\mathsf{cw}} : \mathsf{CW} \to \mathsf{AbGrp}$, but the universe of abelian groups is a groupoid. We may, however, apply the rule for groupoid-valued elimination of propositional truncations [Kra15, Proposition 3]. Applied to the goal in question, it says that we may define H_n^{cw} by (1) defining $H_n^{\mathsf{cw}}(X_\infty)$ for CW-skeleta $X_\bullet$, (2) showing that for $e : X_\infty \simeq Y_\infty$, we have an isomorphism $e_* : H_n^{\mathsf{cw}}(X_\infty) \cong H_n^{\mathsf{cw}}(Y_\infty)$ and (3) that e_* is functorial. For (1), we simply set $H_n^{\mathsf{cw}}(X_\infty) := H_n^{\mathsf{skel}_\infty}(X_\bullet)$. The conditions (2) and (3) follow from functoriality of $H_n^{\mathsf{skel}_\infty}$. Functoriality of H_n^{cw} follows in a similar manner. This completes the definition of the cellular homology functors H_n^{cw}.

The formalisation of the analoguous cohomology theory as well as the verification of the Eilenberg-Steenrod axioms is future/ongoing work. When this is completed, we are planning to compute cellular (co)homology groups of some well-known spaces and use `Cubical Agda` to do concrete computations involving our (co)homology theory. Our hope is that the development of cellular (co)homology will perform better than other alternatives and will be able to compute e.g. some of the examples that failed in [BLM22, Section 6]. We also hope that the results we present here will be useful in the formalisation of recent work by Barton and Campion [Bar22] on a synthetic proof of Serre's finiteness theorem for homotopy groups of spheres, which, in fact, was the orignal motivation behind this project.

[2] Although $\mathsf{CW}^{\mathsf{skel}}$ is wild, AbGrp is a 1-category, and hence functoriality is interpreted in the 1-categorical sense.

References

[Bar22] R. Barton. *Finite presentability of homotopy groups of spheres*. Talk at the Seminar on Homotopy Type Theory at CMU, presenting joint work with Tim Campion. 2022. URL: https://www.cmu.edu/dietrich/philosophy/hott/seminars/previous.html.

[BLM22] G. Brunerie, A. Ljungström, and A. Mörtberg. "Synthetic Integral Cohomology in Cubical Agda". In: *30th EACSL Annual Conference on Computer Science Logic (CSL 2022)*. Ed. by F. Manea and A. Simpson. Vol. 216. Leibniz International Proceedings in Informatics (LIPIcs). Dagstuhl, Germany: Schloss Dagstuhl – Leibniz-Zentrum für Informatik, 2022, 11:1–11:19. ISBN: 978-3-95977-218-1. DOI: 10.4230/LIPIcs.CSL.2022.11.

[BH18] U. Buchholtz and K.-B. Hou Favonia. "Cellular Cohomology in Homotopy Type Theory". In: *Proceedings of the 33rd Annual ACM/IEEE Symposium on Logic in Computer Science*. LICS '18. Oxford, United Kingdom: Association for Computing Machinery, 2018, pp. 521–529. ISBN: 9781450355834. DOI: 10.1145/3209108.3209188.

[Cav15] E. Cavallo. "Synthetic Cohomology in Homotopy Type Theory". MA thesis. Carnegie Mellon University, 2015.

[CS23] J. D. Christensen and L. Scoccola. "The Hurewicz theorem in Homotopy Type Theory". In: *Algebraic & Geometric Topology* 23 (5 2023), pp. 2107–2140.

[Doo18] F. van Doorn. "On the Formalization of Higher Inductive Types and Synthetic Homotopy Theory". PhD thesis. Carnegie Mellon University, May 2018. URL: https://arxiv.org/abs/1808.10690.

[Gra18] R. Graham. *Synthetic Homology in Homotopy Type Theory*. Preprint. 2018. arXiv: 1706.01540 [math.LO].

[Kra15] N. Kraus. "The General Universal Property of the Propositional Truncation". In: *20th International Conference on Types for Proofs and Programs (TYPES 2014)*. Ed. by H. Herbelin, P. Letouzey, and M. Sozeau. Vol. 39. Leibniz International Proceedings in Informatics (LIPIcs). Dagstuhl, Germany: Schloss Dagstuhl – Leibniz-Zentrum für Informatik, 2015, pp. 111–145. ISBN: 978-3-939897-88-0. DOI: 10.4230/LIPIcs.TYPES.2014.111.

[LF14] D. R. Licata and E. Finster. "Eilenberg-MacLane Spaces in Homotopy Type Theory". In: *Proceedings of the Joint Meeting of the Twenty-Third EACSL Annual Conference on Computer Science Logic (CSL) and the Twenty-Ninth Annual ACM/IEEE Symposium on Logic in Computer Science (LICS)*. CSL-LICS '14. Vienna, Austria: Association for Computing Machinery, 2014. ISBN: 9781450328869. DOI: 10.1145/2603088.2603153.

[May99] J. May. *A Concise Course in Algebraic Topology*. Chicago Lectures in Mathematics. University of Chicago Press, 1999. ISBN: 9780226511832.

[Shu13] M. Shulman. *Cohomology*. post on the Homotopy Type Theory blog: http://homotopytypetheory.org/2013/07/24/. 2013.

[Spe18] T. Spectral Sequence Project. https://github.com/cmu-phil/Spectral. 2018.

[VMA21] A. Vezzosi, A. Mörtberg, and A. Abel. "Cubical Agda: A dependently typed programming language with univalence and higher inductive types". In: *Journal of Functional Programming* 31 (2021), e8. DOI: 10.1017/S0956796821000034.