Differential Geometry in Synthetic Algebraic Geometry

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We present some work in progress [3], joint with Felix Cherubini, Matthias Hutzler and David Wärn.

1 Synthetic algebraic geometry

Synthetic algebraic geometry is a recent formal system [2] which consists of HoTT extended by three axioms:

Axiom 1 There is a local ring R.

The underlying type of this ring is assumed to be a set to avoid coherence issues. This means synthetic algebraic geometry is about *non-derived* algebraic geometry.

Given a finitely presented R-algebra A we can define its corresponding affine scheme $\mathrm{Spec}(A)$ by:

$$\mathrm{Spec}(A) = \mathrm{Hom}_{R-\mathrm{Alg}}(A, R)$$

For example, if we have:

$$A = R[X_1, \cdots, X_n]/(P)$$

then we have that $\mathrm{Spec}(A)$ is the type of roots of P, that is:

$$\mathrm{Spec}(A) = \{(x_1, \cdots, x_n) : R^n \mid P(x_1, \cdots, x_n) = 0\}$$

Axiom 2 For any finitely presented algebra A, the map:

$$A \to (\mathrm{Spec}(A) \to R)$$

is an equivalence. So the Spec construction induces an equivalence between finitely presented algebras and affine schemes.

This generalises the usual Kock-Lawvere axiom of synthetic differential geometry from Weil algebras to all finitely presented algebras.

Axiom 3 Affine schemes enjoy a weakening of the axiom of choice called Zariski local choice.

It is conjectured that these axioms can be justified by interpreting HoTT into the higher Zariski topos [5]. An alternative model is given in Section 8 of [2] using internal sheaves, but it is unclear how it relates to the higher Zariski topos.

Algebraic geometry often focuses on studying schemes, which can be defined synthetically as sets that are locally equivalent to affine schemes, i.e. locally defined as sets of roots of polynomials.

Synthetic algebraic geometry then consists of studying these schemes using HoTT and the three aforementioned axioms. Our goal in this talk is to report on our development of differential geometry for these synthetic schemes.

2 Differential geometry of synthetic schemes

A key idea of algebraic geometry is that nilpotent elements can be used to encode infinitesimal information. For example, synthetically:

Definition For X a type and $x : X$, the tangent space of X at x is defined by:

$$T_x(X) = \{t : \mathbb{D} \to X \mid t(0) = x\}$$

where:

$$\mathbb{D} = \{r : R \mid r^2 = 0\}$$

This agrees with the usual definitions when X is a scheme.

For any map $f : X \to Y$ and any $x : X$ we have a differential:

$$df_x : T_x(X) \to T_{f(x)}(Y)$$

defined by post-composing with f.

A proposition P is called closed dense if it is merely of the form:

$$x_1 = 0 \wedge \cdots \wedge x_n = 0$$

for $x_1, \cdots, x_n : R$ nilpotent. Now we can define smoothness:

Definition A type X is smooth if for any closed dense proposition P the map:

$$X \to X^P$$

is surjective.

So when proving that a smooth type is merely inhabited, we are allowed to cancel finitely many nilpotents in R. But what does this have to do with differential geometry? In this talk, we focus on the following two results:

Proposition A map between smooth schemes has smooth fibers if and only if its differentials are surjective.

Proposition Smooth schemes have finite free tangent spaces

In the notes [3], we study formally étale types. They are defined similarly to smooth types, only with surjections replaced by equivalences. Being formally étale is a lex modality [4]. Using this modality with the notion of V-manifold from [1], we can leverage the previous proposition into:

Theorem A scheme is smooth of dimension n if and only if it is a R^n-manifold.

References

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