Relating ordinals in set theory to ordinals in type theory

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Ordinals in set theory

- Def. A set x is transitive if for every $y \in x$ and $z \in y$, we have $z \in x$.
- Def. A set-theoretic ordinal is a transitive set whose elements are all transitive.
- Lemma The elements of a set-theoretic ordinal are again set-theoretic ordinals.
  Thus, a set is a set-theoretic ordinal if and only if it is hereditarily transitive.
- Ex. The sets $\emptyset$, $\{\emptyset\}$ and $\{\emptyset, \{\emptyset\}\}$ are all set-theoretic ordinals, but $\{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}\}$ isn't, as $\{\{\emptyset\}\}$ is a non-transitive member.

Ordinals in homotopy type theory

- In HoTT, a (type-theoretic) ordinal is defined as a type X with a prop-valued binary relation $<$ that is transitive, extensional and wellfounded.
- Extensionality means that we have

$$x = y \iff \forall (u : X).(u < x \iff u < y)$$

It follows that X is an hset.

- Wellfoundedness is defined in terms of accessibility, but is equivalent to the assertion that for every $P : X \to \mathcal{U}$, we have $\Pi(x : X).P(x)$ as soon as $\Pi(x : X).(\Pi(y : X).(y < x \to P(y))) \to P(x)$.

Types of ordinals in HoTT

- We write Ord for the type of (small) type-theoretic ordinals.
- HoTT hosts a model $(\mathbb{V}, \in)$ of a constructive set theory.

  The type $\mathbb{V}$ is a HIT with point-constructor

  $$\mathbb{V}\text{-set}(A, f) : \mathbb{V} \quad \text{for} \quad A : \mathcal{U} \text{ and } f : A \to \mathbb{V}$$

  quotiented by bisimilarity: $\mathbb{V}\text{-set}(A, f)$ and $\mathbb{V}\text{-set}(B, g)$ are identified exactly when f and g have the same image.
- We define set-membership $\in : \mathbb{V} \to \mathbb{V} \to \mathsf{Prop}$ by

  $$x \in \mathbb{V}\text{-set}(A, f) :\equiv \exists (a : A). f(a) = x$$
- Thus, we can define the subtype $\mathbb{V}_{\mathrm{ord}}$ of $\mathbb{V}$ of set-theoretic ordinals in HoTT.

Set-theoretic and type-theoretic ordinals are equivalent

- Thm. The types $\mathbb{V}_{\mathsf{ord}}$ and Ord are equivalent.
- Proof sketch Define $\Phi : \mathsf{Ord} \to \mathbb{V}_{\mathsf{ord}}$ by transfinite recursion:

$$\Phi(\alpha) :\equiv \mathbb{V}\text{-}\mathsf{set}(\alpha, \lambda(a : \alpha).\Phi(\alpha \downarrow a)),$$

where

$$\alpha \downarrow a \;:\equiv\; \Sigma(b : \alpha).b < a.$$

Its inverse $\Psi : \mathbb{V}_{\mathsf{ord}} \to \mathsf{Ord}$ is the rank function:

$$\Psi(\mathbb{V}\text{-}\mathsf{set}(A, f)) :\equiv \bigvee_{a:A} (\Psi(f(a)) + \mathbf{1}).$$

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- It is possible to give nonrecursive descriptions of the rank:

$$\Psi(x) \simeq \Sigma(y : \mathbb{V}_{\mathsf{ord}}).y \in x \quad \text{and} \quad \Psi(\mathbb{V}\text{-set}(A, f)) = A/{\sim},$$

where $a \sim b \iff f(a) = f(b)$. (But be careful about size.)

The big picture

- Thm. The types $\mathbb{V}_{\mathsf{ord}}$ and Ord are equivalent.

  But more is true...

- The type Ord is actually a large type-theoretic ordinal itself:

$$\begin{aligned} \alpha \prec \beta &\iff \alpha \text{ is an initial segment of } \beta \\ &\iff \Sigma(y : \beta).(\alpha = \beta \downarrow y) \end{aligned}$$

- Membership $\in$ makes $\mathbb{V}_{\mathsf{ord}}$ into a large type-theoretic ordinal.

- Thm. The type-theoretic ordinals $(\mathbb{V}_{\mathsf{ord}}, \in)$ and $(\mathsf{Ord}, \prec)$ are isomorphic.

  Thus, in HoTT,

  set-theoretic and type-theoretic ordinals coincide.

The *bigger* picture

- Can we realize the *full* cumulative hierarchy $\mathbb{V}$ as a type of ordered structures?
  That is, can we find a type making the square

$$\begin{tikzcd} \mathbb{V}_{\mathsf{ord}} \arrow[r, "\simeq"] \arrow[d, hook] & \mathsf{Ord} \arrow[d, hook] \\ \mathbb{V} \arrow[r, "\simeq"] & ? \end{tikzcd}$$

  commute?

- An initial naive attempt may be to simply drop transitivity, i.e., to take

$$? = \text{type of extensional wellfounded orders}.$$

Generalizing from ordinals to sets

- We consider extensional wellfounded orders $(X, <)$ with a marking: a predicate on X that picks out the top-level elements of a set.
- E.g., the sets $\{\emptyset, \{\emptyset\}\}$ and $\{\{\emptyset\}\}$ are both represented by the two-element type ordered as $0 < 1$; we mark both 0 and 1 for the first set, but only 1 in the representation of the second set.
- A marking is covering if any element can be reached from a marked top-level element, i.e., if the order contains no "junk".
- The idea of encoding sets as wellfounded structures isn't new. The above approach worked well for our purposes of generalizing the theory of ordinals.

Filling the bigger picture

- We write $\mathsf{MEWO}_{\mathsf{cov}}$ for the type of covered marked extensional wellfounded orders.
- Every ordinal can be equipped with the trivial covering by marking everything. Thus, the type Ord of ordinals is a subtype of $\mathsf{MEWO}_{\mathsf{cov}}$.
- We get the bottom isomorphism by generalizing the constructions used to establish $\mathbb{V}_{\mathsf{ord}} \simeq \mathsf{Ord}$:

$$
\begin{tikzcd}
\mathbb{V}_{\mathsf{ord}} \arrow[r, "\simeq"] \arrow[d, hook] & \mathsf{Ord} \arrow[d, hook] \\
\mathbb{V} \arrow[r, "\simeq"] & \mathsf{MEWO}_{\mathsf{cov}}
\end{tikzcd}
$$

Conclusion

- In HoTT, the set-theoretic ordinals in $\mathbb{V}$ coincide with the type-theoretic ordinals.
- By generalizing from type-theoretic ordinals to covered mewos, we capture all sets in $\mathbb{V}$.
- Question: Do the type-theoretic ordinals in the cubical sets model of HoTT coincide with the set-theoretic ordinals?
- Question: Can we use covered mewos to pin down the exact constructive set theory that $\mathbb{V}$ models? E.g., can we show strong collection is independent?
- *Set-Theoretic and Type-Theoretic Ordinals Coincide*. TdJ, Nicolai Kraus, Fredrik Nordvall Forsberg and Chuangjie Xu. `arXiv:2301.10696`. Accepted for presentation at *LICS'23*. Fully formalized in AGDA.