Relating ordinals in set theory to ordinals in type theory

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 Ordinals in set theory

- **Def.** A set \( x \) is **transitive** if for every \( y \in x \) and \( z \in y \), we have \( z \in x \).

- **Def.** A **set-theoretic ordinal** is a transitive set whose elements are all transitive.

- **Lemma** The elements of a set-theoretic ordinal are again set-theoretic ordinals. Thus, a set is a set-theoretic ordinal if and only if it is **hereditarily transitive**.

- **Ex.** The sets \( \emptyset, \{\emptyset\} \) and \( \{\emptyset, \{\emptyset\}\} \) are all set-theoretic ordinals, but \( \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}\} \) isn’t, as \( \{\{\emptyset\}\} \) is a non-transitive member.
Ordinals in homotopy type theory

- In HoTT, a \textit{(type-theoretic) ordinal} is defined as a type $X$ with a prop-valued binary relation $<$ that is \textit{transitive}, \textit{extensional} and \textit{wellfounded}.

- Extensionality means that we have

  $$x = y \iff \forall (u : X).(u < x \iff u < y)$$

  It follows that $X$ is an hset.

- Wellfoundedness is defined in terms of \textit{accessibility}, but is equivalent to the assertion that for every $P : X \to \mathcal{U}$, we have

  $$\prod(x : X).P(x) \text{ as soon as } \prod(x : X).((\prod(y : X).(y < x \to P(y))) \to P(x)).$$
Types of ordinals in HoTT

- We write $\text{Ord}$ for the type of (small) type-theoretic ordinals.

- HoTT hosts a model $(V, \in)$ of a constructive set theory. The type $V$ is a HIT with point-constructor

  $$V\text{-set}(A, f) : V \quad \text{for} \quad A : U \text{ and } f : A \to V$$

  quotiented by bisimilarity: $V\text{-set}(A, f)$ and $V\text{-set}(B, g)$ are identified exactly when $f$ and $g$ have the same image.

- We define set-membership $\in : V \to V \to \text{Prop}$ by

  $$x \in V\text{-set}(A, f) \equiv \exists (a : A). f(a) = x$$

- Thus, we can define the subtype $V_{\text{ord}}$ of $V$ of set-theoretic ordinals in HoTT.
Set-theoretic and type-theoretic ordinals are equivalent

Thm. The types $\mathbb{V}_{\text{ord}}$ and $\text{Ord}$ are equivalent.

Proof sketch Define $\Phi : \text{Ord} \to \mathbb{V}_{\text{ord}}$ by transfinite recursion:

$$\Phi(\alpha) \equiv \mathbb{V}\text{-set}(\alpha, \lambda(a : \alpha).\Phi(\alpha \downarrow a)),$$

where

$$\alpha \downarrow a \equiv \Sigma(b : \alpha).b < a.$$

Its inverse $\Psi : \mathbb{V}_{\text{ord}} \to \text{Ord}$ is the rank function:

$$\Psi(\mathbb{V}\text{-set}(A, f)) \equiv \bigvee_{a : A} (\Psi(f(a)) + 1).$$
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It is possible to give nonrecursive descriptions of the rank:

$$\Psi(x) \simeq \Sigma(y : \mathbb{V}_{\text{ord}}). y \in x \quad \text{and} \quad \Psi(\mathbb{V}\text{-set}(A, f)) = A/\sim,$$

where $a \sim b \iff f(a) = f(b)$. (But be careful about size.)
The big picture

▶ **Thm.** The types $\mathbb{V}_{\text{ord}}$ and $\text{Ord}$ are equivalent.

But more is true...

▶ The type $\text{Ord}$ is actually a large type-theoretic ordinal itself:

$$\alpha < \beta \iff \alpha \text{ is an initial segment of } \beta \iff \sum(y : \beta). (\alpha = \beta \downarrow y)$$

▶ Membership $\in$ makes $\mathbb{V}_{\text{ord}}$ into a large type-theoretic ordinal.

▶ **Thm.** The type-theoretic ordinals $(\mathbb{V}_{\text{ord}}, \in)$ and $(\text{Ord}, <)$ are isomorphic.

Thus, in HoTT, set-theoretic and type-theoretic ordinals coincide.
**The bigger picture**

▶ Can we realize the *full* cumulative hierarchy $\bigvee$ as a type of ordered structures? That is, can we find a type making the square commute?

\[
\begin{array}{c}
\bigvee_{\text{ord}} \xrightarrow{\sim} \text{Ord} \\
\downarrow \quad \downarrow \\
\bigvee \xrightarrow{\sim} ?
\end{array}
\]

▶ An initial naive attempt may be to simply *drop transitivity*, i.e., to take

\[? = \text{type of extensional wellfounded orders.}\]
Generalizing from ordinals to sets

- We consider extensional wellfounded orders \((X, <)\) with a marking: a predicate on \(X\) that picks out the top-level elements of a set.

- E.g., the sets \(\{\emptyset, \{\emptyset\}\}\) and \(\{\{\emptyset\}\}\) are both represented by the two-element type ordered as \(0 < 1\); we mark both 0 and 1 for the first set, but only 1 in the representation of the second set.

- A marking is covering if any element can be reached from a marked top-level element, i.e., if the order contains no “junk”.

- The idea of encoding sets as wellfounded structures isn’t new. The above approach worked well for our purposes of generalizing the theory of ordinals.
Filling the bigger picture

- We write $\text{MEWO}_{\text{cov}}$ for the type of covered marked extensional wellfounded orders.

- Every ordinal can be equipped with the trivial covering by marking everything. Thus, the type $\text{Ord}$ of ordinals is a subtype of $\text{MEWO}_{\text{cov}}$.

- We get the bottom isomorphism by generalizing the constructions used to establish $\mathbb{V}_{\text{ord}} \simeq \text{Ord}$:

\[
\begin{array}{ccc}
\mathbb{V}_{\text{ord}} & \xrightarrow{\simeq} & \text{Ord} \\
\downarrow & & \downarrow \\
\mathbb{V} & \xrightarrow{\simeq} & \text{MEWO}_{\text{cov}}
\end{array}
\]
Conclusion

▶ In HoTT, the set-theoretic ordinals in \( \mathbb{V} \) coincide with the type-theoretic ordinals.

▶ By generalizing from type-theoretic ordinals to covered mewos, we capture all sets in \( \mathbb{V} \).

▶ Question: Do the type-theoretic ordinals in the cubical sets model of HoTT coincide with the set-theoretic ordinals?

▶ Question: Can we use covered mewos to pin down the exact constructive set theory that \( \mathbb{V} \) models? E.g., can we show strong collection is independent?