Rezk completion of bicategories

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1. Rezk completion of categories

2. Rezk completion of bicategories
    - Towards computing Rezk completions

3. Open question

Rezk completion

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Rezk completion

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2. Any functor $F : \mathcal{C} \to \mathcal{E}$ with $\mathcal{E}$ univalent, factors *uniquely* via $\mathcal{H}$:

$$\begin{tikzcd} \mathcal{C} \arrow[d, "\mathcal{H}"'] \arrow[dr, "F"] & \\ \mathrm{RC}(\mathcal{C}) \arrow[r, dashed, "\exists!"'] & \mathcal{E} \end{tikzcd}$$

Rezk completion: Definition

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A **Rezk completion** of a category $\mathcal{C}$ consists of:

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such that for any univalent category $\mathcal{E}$,

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Remark

1. Equivalently: $\mathcal{H} \cdot (-)$ is adjoint equivalence of categories.
2. Equivalently: $\mathcal{H} \cdot (-)$ is weak equivalence of categories.

Rezk completion of bicategories

Bicategories in Univalent Foundations

For every locally univalent bicategory $\mathcal{B}$, there is a univalent bicategory $\mathrm{RC}_{\mathrm{global}}(\mathcal{B})$ and a weak equivalence $\mathcal{B} \to \mathrm{RC}_{\mathrm{global}}(\mathcal{B})$.

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What about non-locally univalent bicategories:

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Theorem

For every bicategory $\mathcal{B}$, there is a locally univalent bicategory $\mathsf{RC}_{\mathsf{local}}(\mathcal{B})$ and a weak equivalence $\mathcal{B} \to \mathsf{RC}_{\mathsf{local}}(\mathcal{B})$.

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Define

1. $\mathsf{ob}\left(\mathsf{RC}_{\mathsf{local}}(\mathcal{B})\right) := \mathsf{ob}\left(\mathcal{B}\right)$;
2. $\mathsf{RC}_{\mathsf{local}}(\mathcal{B})\,(x, y) := \mathsf{RC}\left(\mathcal{B}(x, y)\right)$

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3. What about composition?

Rezk completion of bicategories: Composition

$$\begin{tikzcd}
\mathcal{B}(x,y) \times \mathcal{B}(y,z) \arrow[r, "\eta \times \eta"] \arrow[d, "\cdot_{\mathcal{B}}"'] & \mathsf{RC}_{\mathsf{local}}(\mathcal{B}(x,y)) \times \mathsf{RC}_{\mathsf{local}}(\mathcal{B}(y,z)) \arrow[d, "\exists!"] \\
\mathcal{B}(x,z) \arrow[r, "\eta"'] & \mathsf{RC}_{\mathsf{local}}(\mathcal{B}(x,z))
\end{tikzcd}$$

Rezk completion of bicategories: Left unitor

$$
\begin{tikzcd}
\mathcal{B}(x,y) \arrow[r, "\eta"] \arrow[d, "(\mathrm{Id},-)"'] \arrow[dd, bend right, "\mathrm{Id}"'] & \mathrm{RC}_{\mathrm{local}}(\mathcal{B}(x,y)) \arrow[d, "(\mathrm{Id},-)"] \\
\mathcal{B}(x,x) \times \mathcal{B}(x,y) \arrow[r, "\eta\times\eta"] \arrow[d, "\cdot_{\mathcal{B}}"'] & RC_{\mathrm{local}}(\mathcal{B}(x,x)) \times \mathrm{RC}_{\mathrm{local}}(\mathcal{B}(x,y)) \arrow[d, "\cdot_{\mathrm{RC}_{\mathrm{local}}(\mathcal{B})}"] \\
\mathcal{B}(x,y) \arrow[r, "\eta"'] & \mathrm{RC}_{\mathrm{local}}(\mathcal{B}(x,y))
\end{tikzcd}
$$

Rezk completion of bicategories: universal property

Let $\mathcal{B}$ be a bicategory and $\eta_{\mathcal{B}} : \mathcal{B} \to \mathsf{RC}_2(\mathcal{B})$ be the Rezk completion of $\mathcal{B}$.

Conjecture

For any univalent bicategory $\mathcal{D}$, the pseudo-functor

$$\eta_{\mathcal{B}} \cdot - : [\mathsf{RC}_2(\mathcal{B}), \mathcal{D}] \to [\mathcal{B}, \mathcal{D}]$$

is a bi-equivalence of bicategories. Furthermore, this characterizes the $(\mathsf{RC}_2(\mathcal{B}), \eta_{\mathcal{B}})$ uniquely.

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Rezk completion of **Cat**(?)

Question

What is the Rezk completion $\mathrm{RC}_2(\mathbf{Cat})$ of $\mathbf{Cat}$?

Rezk completion of $\mathbf{Cat}$: possible approach 1

Observation

Consider the following (commuting) diagram:

$$\begin{tikzcd}
\mathbf{Cat} \arrow[rr, "\eta_{\mathbf{Cat}}"] \arrow[dr, "\mathsf{RC}_1"'] & & \mathsf{RC}_2(\mathbf{Cat}) \arrow[dl, "\exists!"] \\
& \mathbf{Cat}_{\mathsf{univ}} &
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Thus $\mathsf{RC}_2(\mathbf{Cat})$ can be constructed as a displayed bicategory over $\mathbf{Cat}_{\mathsf{univ}}$, i.e. as a bicategory of structured (univalent) categories.

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Thus $\mathsf{RC}_2(\mathbf{Cat})$ can be constructed as a displayed bicategory over $\mathbf{Cat}_{\mathsf{univ}}$, i.e. as a bicategory of structured (univalent) categories.

Question

Can we construct this displayed bicategory concretely?

Rezk completion of **Cat**: possible approach 2

Another approach is:

Question

1. What is the *local* Rezk completion $\mathsf{RC}_{\mathsf{local}}(\mathbf{Cat})$ of **Cat**?
2. What is the Rezk completion $\mathsf{RC}_1([\mathcal{C}, \mathcal{D}])$ of a functor category $[\mathcal{C}, \mathcal{D}]$?

Understanding the Rezk completion

Proposition

For every $\mathcal{C}$ and $\mathcal{D}$, TFAE;

1. $\mathrm{RC}([\mathcal{C}, \mathcal{D}]) = [\mathcal{C}, \mathrm{RC}(\mathcal{D})]$;
2. $\mathcal{C}$ and $\mathcal{D}$ are equivalent to a univalent category.

Understanding the Rezk completion

Proposition

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Lemma

Let $\mathcal{B}$ be a full sub-bicategory of **Cat**. TFAE:

1. $\prod_{\mathcal{C},\mathcal{D}:\mathcal{B}} \mathrm{RC}_1([\mathcal{C}, \mathcal{D}]) = [\mathcal{C}, \mathrm{RC}_1(\mathcal{D})]$;
2. $\mathrm{RC}_2(\mathcal{B}) = \mathcal{B}_{\mathrm{univ}}$ (where $\mathcal{B}_{\mathrm{univ}}$ is the intersection of $\mathcal{B}$ and $\mathbf{Cat}_{\mathrm{univ}}$) ;
3. $\mathcal{B} \simeq \mathcal{B}_{\mathrm{univ}}$.

Rezk completion of **Cat**

Corollary

$\mathsf{RC}(\mathbf{Cat}) \neq \mathbf{Cat}_{\mathsf{univ}}$.

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Intuition 1

1. $\mathbf{Cat}_{\mathrm{univ}} \sim$ categories up to weak equivalence (unique representing object) ;
2. $\mathrm{RC}(\mathbf{Cat}) \sim$ categories up to isomorphism of isomorphic functors.

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Corollary

$\mathrm{RC}(\mathbf{Cat}) \neq \mathbf{Cat}_{\mathrm{univ}}$.

Intuition 1

1. $\mathbf{Cat}_{\mathrm{univ}} \sim$ categories up to weak equivalence (unique representing object) ;
2. $\mathrm{RC}(\mathbf{Cat}) \sim$ categories up to isomorphism of isomorphic functors.

Intuition 2

1. $\mathrm{RC} : \mathbf{Cat} \to \mathbf{Cat}_{\mathrm{univ}}$ is *objectwise* free;
2. $\eta_{\mathcal{C}} : \mathbf{Cat} \to \mathrm{RC}(\mathbf{Cat})$ is (externally) free

Is the Rezk completion of categories what we want?

Intuitively

1. *good* categories $\sim$ univalent categories.

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Is there a completion GC on bicategories such that

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Question

Is there a completion GC on bicategories such that

1. $\text{GC}(\mathbf{Cat}) = \mathbf{Cat}_{\text{univ}}$;
2. $\text{GC}(\mathbf{MonCat}) = \mathbf{MonCat}_{\text{univ}}$;
3. $\text{GC}(\mathbf{DagCat}) = \mathbf{DagCat}_{\dagger-\text{univ}}$?

What does such a completion mean?

Question

Is it possible (universally) characterize $\mathrm{RC} : \mathbf{Cat} \to \mathbf{Cat}_{\mathrm{univ}}$?

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This question is closely related to:

Question

Is it possible to characterize the *correct* notion of univalence of (structured) categories?

The end

Thank you! Any questions?