Rezk completion of bicategories

Kobe Wullaert

Delft University of Technology

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1 Rezk completion of categories

Rezk completion of bicategories Towards computing Rezk completions

Open question

Rezk completion

A Rezk completion RC(C) of a category C is the free univalent category associated to it.

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Rezk completion

- A Rezk completion RC(C) of a category C is the free univalent category associated to it.
- ② Any functor $F : \mathcal{C} \to \mathcal{E}$ with \mathcal{E} univalent, factors *uniquely* via \mathcal{H} :



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Rezk completion: Definition

Definition

- A **Rezk completion** of a category \mathcal{C} consists of:
 - **1** a univalent category $\mathsf{RC}(\mathcal{C})$;
 - 2 a functor $\mathcal{H}: \mathcal{C} \to \mathsf{RC}(\mathcal{C})$

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1 a univalent category $\mathsf{RC}(\mathcal{C})$;

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such that for any univalent category $\ensuremath{\mathcal{E}}$,

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\mathcal{H} \cdot (-) : [\mathsf{RC}(\mathcal{C}), \mathcal{E}] \to [\mathcal{C}, \mathcal{E}],
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$$\mathcal{H} \cdot (-) : [\mathsf{RC}(\mathcal{C}), \mathcal{E}] \to [\mathcal{C}, \mathcal{E}],$$

is an isomorphism of categories.

Remark

• Equivalently: $\mathcal{H} \cdot (-)$ is adjoint equivalence of categories.

2 Equivalently: $\mathcal{H} \cdot (-)$ is weak equivalence of categories.

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Rezk completion of bicategories

Bicategories in Univalent Foundations

For every locally univalent bicategory \mathcal{B} , there is a univalent bicategory $\mathsf{RC}_{\mathsf{global}}(\mathcal{B})$ and a weak equivalence $\mathcal{B} \to \mathsf{RC}_{\mathsf{global}}(\mathcal{B})$.

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Rezk completion of bicategories

Bicategories in Univalent Foundations

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What about non-locally univalent bicategories:

Rezk completion of bicategories

Bicategories in Univalent Foundations

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What about non-locally univalent bicategories:

Theorem

For every bicategory \mathcal{B} , there is a locally univalent bicategory $\mathsf{RC}_{\mathsf{local}}(\mathcal{B})$ and a weak equivalence $\mathcal{B} \to \mathsf{RC}_{\mathsf{local}}(\mathcal{B})$.

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Rezk completion of bicategories: Construction

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Theorem

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Define

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$$ob(RC_{local}(\mathcal{B})) := ob(\mathcal{B});$$

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Rezk completion of bicategories: Construction

Theorem

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Define

- $ob(RC_{local}(\mathcal{B})) := ob(\mathcal{B});$
- What about composition?

Rezk completion of bicategories: Composition

$$\begin{array}{ccc} \mathcal{B}(x,y) \times \mathcal{B}(y,z) \xrightarrow{\eta \times \eta} \mathsf{RC}_{\mathsf{local}}(\mathcal{B}(x,y)) \times \mathsf{RC}_{\mathsf{local}}(\mathcal{B}(y,z)) \\ & & \downarrow^{\exists !} \\ & \mathcal{B}(x,z) \xrightarrow{\eta} & \mathsf{RC}_{\mathsf{local}}(\mathcal{B}(x,z)) \end{array}$$

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Rezk completion of bicategories: Left unitor

$$\mathsf{Id} \underbrace{\begin{array}{c} \mathcal{B}(x,y) \xrightarrow{\eta} & \mathsf{RC}_{\mathsf{local}}(\mathcal{B}(x,y)) \\ (\mathsf{Id},-) \downarrow & \downarrow(\mathsf{Id},-) \\ \mathcal{B}(x,x) \times \mathcal{B}(x,y) \xrightarrow{\eta \times \eta} & \mathsf{RC}_{\mathsf{local}}(\mathcal{B}(x,x)) \times \mathsf{RC}_{\mathsf{local}}(\mathcal{B}(x,y)) \\ & & \downarrow^{\cdot_{\mathsf{RC}_{\mathsf{local}}(\mathcal{B})} \\ & & \downarrow^{\cdot_{\mathsf{RC}_{\mathsf{local}}(\mathcal{B})} \\ & & & \downarrow^{\cdot_{\mathsf{RC}_{\mathsf{local}}(\mathcal{B})} \\ & & & & \downarrow^{\cdot_{\mathsf{RC}}(\mathcal{B})} \\ & & & & \mathsf{RC}_{\mathsf{local}}(\mathcal{B}(x,y)) \end{array}}$$

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Rezk completion of bicategories: universal property

Let \mathcal{B} be a bicategory and $\eta_{\mathcal{B}} : \mathcal{B} \to \mathsf{RC}_2(\mathcal{B})$ be the Rezk completion of \mathcal{B} .

Conjecture

For any univalent bicategory \mathcal{D}_{r} the pseudo-functor

$$\eta_{\mathcal{B}} \cdot - : [\mathsf{RC}_2(\mathcal{B}), \mathcal{D}] \to [\mathcal{B}, \mathcal{D}]$$

is a bi-equivalence of bicategories. Furthermore, this characterizes the $({\rm RC}_2(\mathcal{B}),\eta_{\mathcal{B}})$ uniquely.



Rezk completion of bicategories Towards computing Rezk completions

3 Open question

Towards computing Rezk completions

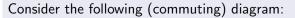
Rezk completion of **Cat**(?)

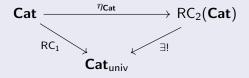
Question

What is the Rezk completion $RC_2(Cat)$ of Cat?

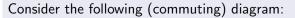
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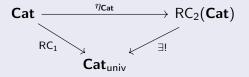
Observation





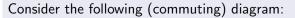
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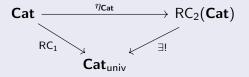




Thus $RC_2(Cat)$ can be constructed as a displayed bicategory over Cat_{univ} , i.e. as a bicategory of structured (univalent) categories.

Observation





Thus $RC_2(Cat)$ can be constructed as a displayed bicategory over Cat_{univ} , i.e. as a bicategory of structured (univalent) categories.

Question

Can we construct this displayed bicategory concretely?

Another approach is:

Question

- What is the *local* Rezk completion RC_{local}(Cat) of Cat?
- What is the Rezk completion RC₁([C, D]) of a functor category [C, D]?

Understanding the Rezk completion

Proposition

- For every C and D, TFAE;
 - $\mathsf{RC}([\mathcal{C}, \mathcal{D}]) = [\mathcal{C}, \mathsf{RC}(\mathcal{D})];$
 - 2 $\mathcal C$ and $\mathcal D$ are equivalent to a univalent category.

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Understanding the Rezk completion

Proposition

For every C and D, TFAE;

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Lemma

Let \mathcal{B} be a full sub-bicategory of **Cat**. TFAE:

- $\label{eq:RC2} \textbf{RC}_2(\mathcal{B}) = \mathcal{B}_{univ} \mbox{ (where } \mathcal{B}_{univ} \mbox{ is the intersection of } \mathcal{B} \mbox{ and } \\ \textbf{Cat}_{univ}) \mbox{ ;}$
- $\ \, \mathfrak{B}\simeq \mathcal{B}_{univ}.$

Towards computing Rezk completions

Rezk completion of Cat

Corollary

 $\mathsf{RC}(\mathbf{Cat}) \neq \mathbf{Cat}_{\mathsf{univ}}.$

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Rezk completion of Cat

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 $\mathsf{RC}(\mathbf{Cat}) \neq \mathbf{Cat}_{\mathsf{univ}}.$

Intuition 1

- Cat_{univ} \sim categories up to weak equivalence (unique representing object) ;
- RC(Cat) ~ categories up to isomorphism of isomorphic functors.

Rezk completion of Cat

Corollary

 $\mathsf{RC}(\mathbf{Cat}) \neq \mathbf{Cat}_{\mathsf{univ}}.$

Intuition 1

- Cat_{univ} \sim categories up to weak equivalence (unique representing object) ;
- RC(Cat) ~ categories up to isomorphism of isomorphic functors.

Intuition 2

- **1** RC : Cat \rightarrow Cat_{univ} is *objectwise* free;
- $\ \, \textbf{0} \ \, \eta_{\mathcal{C}}: \mathbf{Cat} \to \mathsf{RC}(\mathbf{Cat}) \text{ is (externally) free}$

Rezk completion of categories

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Open question

Is the Rezk completion of categories what we want?

Intuitively

• good categories \sim univalent categories.

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Is the Rezk completion of categories what we want?

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- **(**) good categories \sim univalent categories.
- **2** Cat \mapsto Cat_{univ}

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- good categories \sim univalent categories.
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 $\mathsf{GoodReplacement}(\mathbf{Cat}) \sim \mathbf{Cat}_{\mathsf{univ}}$

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Question

Is there a completion GC on bicategories such that

$$GC(Cat) = Cat_{univ}$$

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Is the Rezk completion of categories what we want?

Intuitively

- good categories \sim univalent categories.
- $\textbf{2} \quad \textbf{Cat} \mapsto \textbf{Cat}_{univ}$

 $\mathsf{GoodReplacement}(\mathbf{Cat}) \sim \mathbf{Cat}_{\mathsf{univ}}$

Question

Is there a completion GC on bicategories such that

- **2** GC(MonCat) = MonCat_{univ};

3
$$GC(DagCat) = DagCat_{\dagger-univ}$$
?

What does such a completion mean?

Question

Is it possible (universally) characterize $\mathsf{RC}: \mathbf{Cat} \rightarrow \mathbf{Cat}_{\mathsf{univ}}$?

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What does such a completion mean?

Question

Is it possible (universally) characterize $RC : Cat \rightarrow Cat_{univ}$?

This question is closely related to:

Question

Is it possible to characterize the *correct* notion of univalence of (structured) categories?

The end

Thank you! Any questions?

Kobe Wullaert Rezk completion of bicategories

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