# Classification of Covering Spaces and Canonical Change of Basepoint

HoTT/UF Workshop 2023

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# Motivation

learn how to use HoTT as synthetic framework

# Starting point: formalize an exercise from Hatcher

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Concepts involved

- Closed orientable genus $g$ surface
- Degree
- Covering spaces
- Homology

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How to translate classical statements into Homotopy Type Theory …

… such that they are easy to formalize?

# Classification of Covering Spaces

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*already shown in [Buchholtz, Van Doorn, Rijke (2018)]

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Given a space $X$, a **covering space** of $X$ consists of a space $\tilde{X}$ and a map $p : \tilde{X} \to X$ satisfying the following condition:

(∗) For each point $x \in X$ there is an open neighborhood $U$ of $x$ in $X$ such that $p^{-1}(U)$ is a union of disjoint open sets each of which is mapped homeomorphically onto $U$ by $p$.

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- **No need** to track local, point-set topological properties
- Work directly with the **fibers** $p^{-1}(x)$ as family of sets

# Classification

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For a connected, pointed type $(X, x_0)$

$$F \longleftrightarrow H$$

pointed, connected
covering space of $X$

subgroup $\pi_1(X, x_0)$

i.e. predicate $\pi_1(X, x_0) \to \mathrm{hProp}$
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From covering space to subgroup

$F \quad \longmapsto \quad H_F,$ loops $p$ in $X$ for which there exists a loop in the covering space lying over $p$

- **Surjective** via the **universal covering space**
- **Injective** via the **lifting criterion**

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# Lifting Criterion in HoTT

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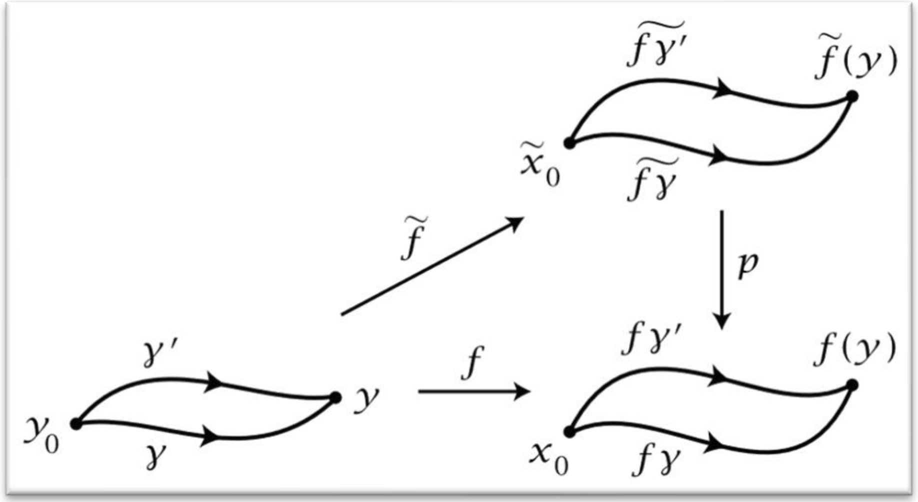
**Proposition 1.33.** *Suppose given a covering space* $p:(\tilde{X},\tilde{x}_0)\to(X,x_0)$ *and a map* $f:(Y,y_0)\to(X,x_0)$ *with* $Y$ *path-connected and locally path-connected. Then a lift* $\tilde{f}:(Y,y_0)\to(\tilde{X},\tilde{x}_0)$ *of* $f$ *exists iff* $f_*(\pi_1(Y,y_0))\subset p_*(\pi_1(\tilde{X},\tilde{x}_0))$.



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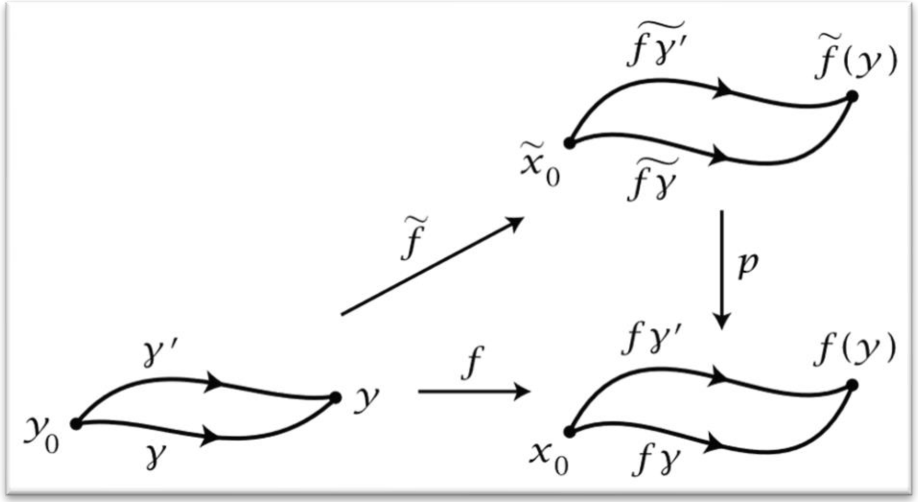


Definitions needed in HoTT

- pointed covering space
- total space and the covering map
- lift of a pointed map to the covering space

Definitions in HoTT

- **Pointed covering space** of $(X, x_0)$

  family $F : X \to \mathrm{hSet}$

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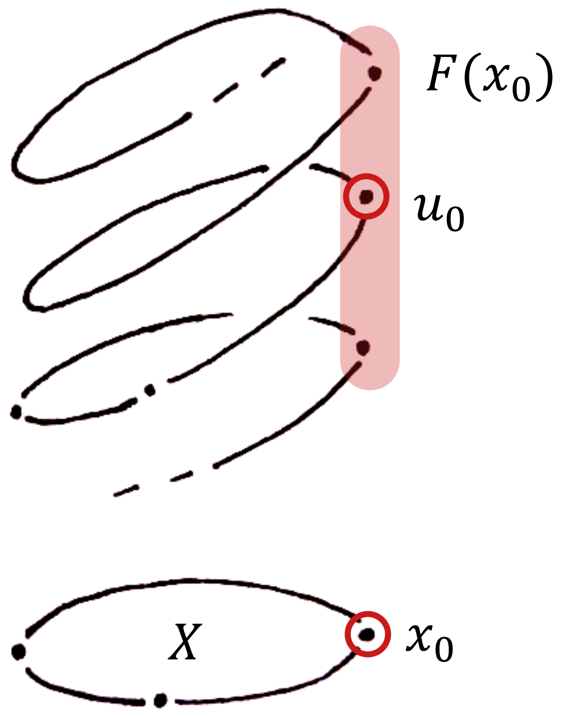

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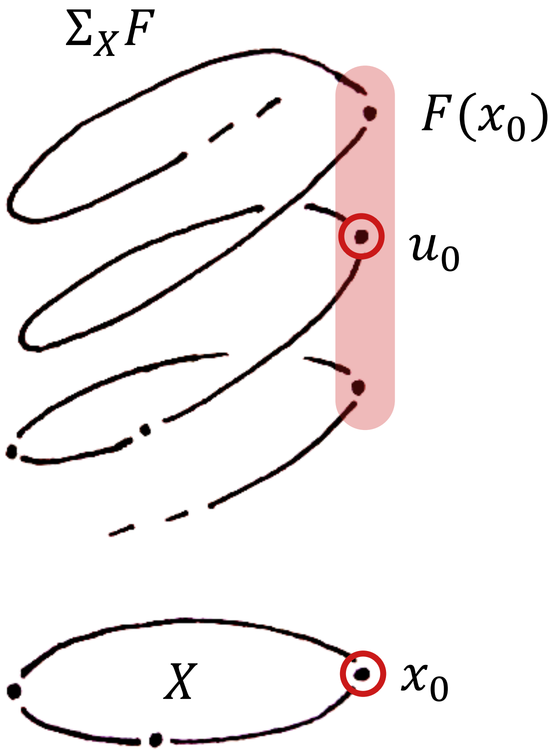

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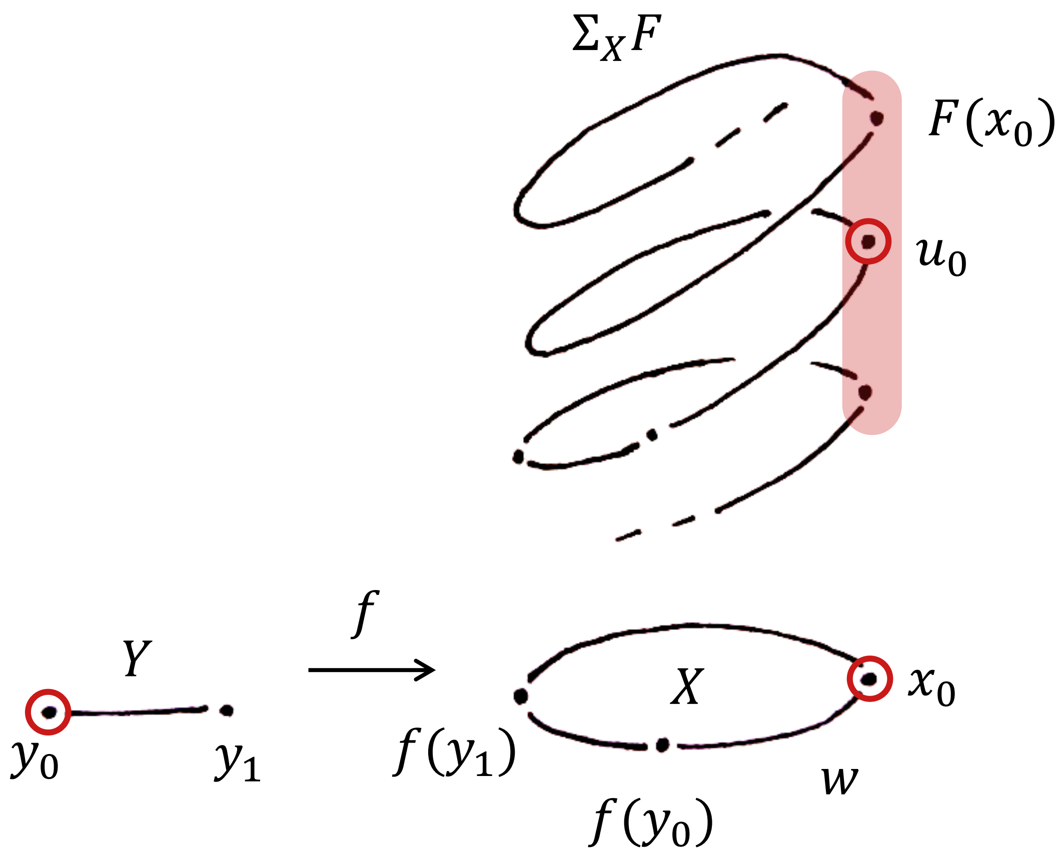

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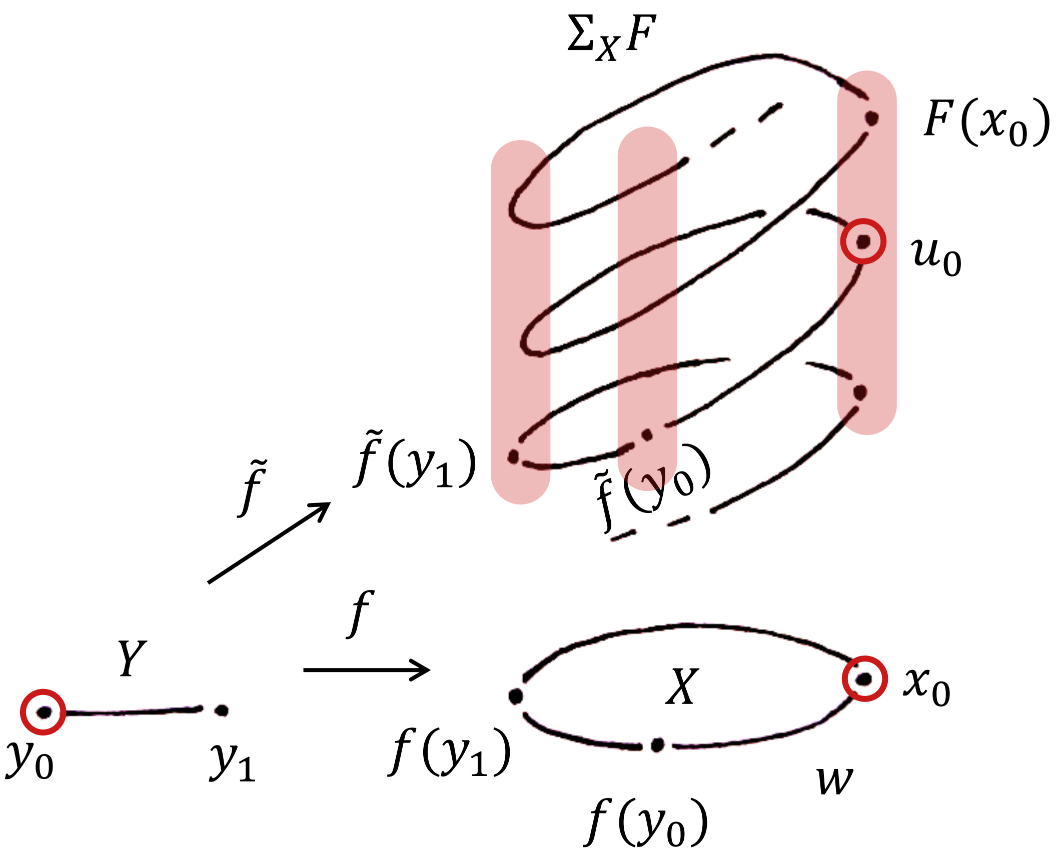

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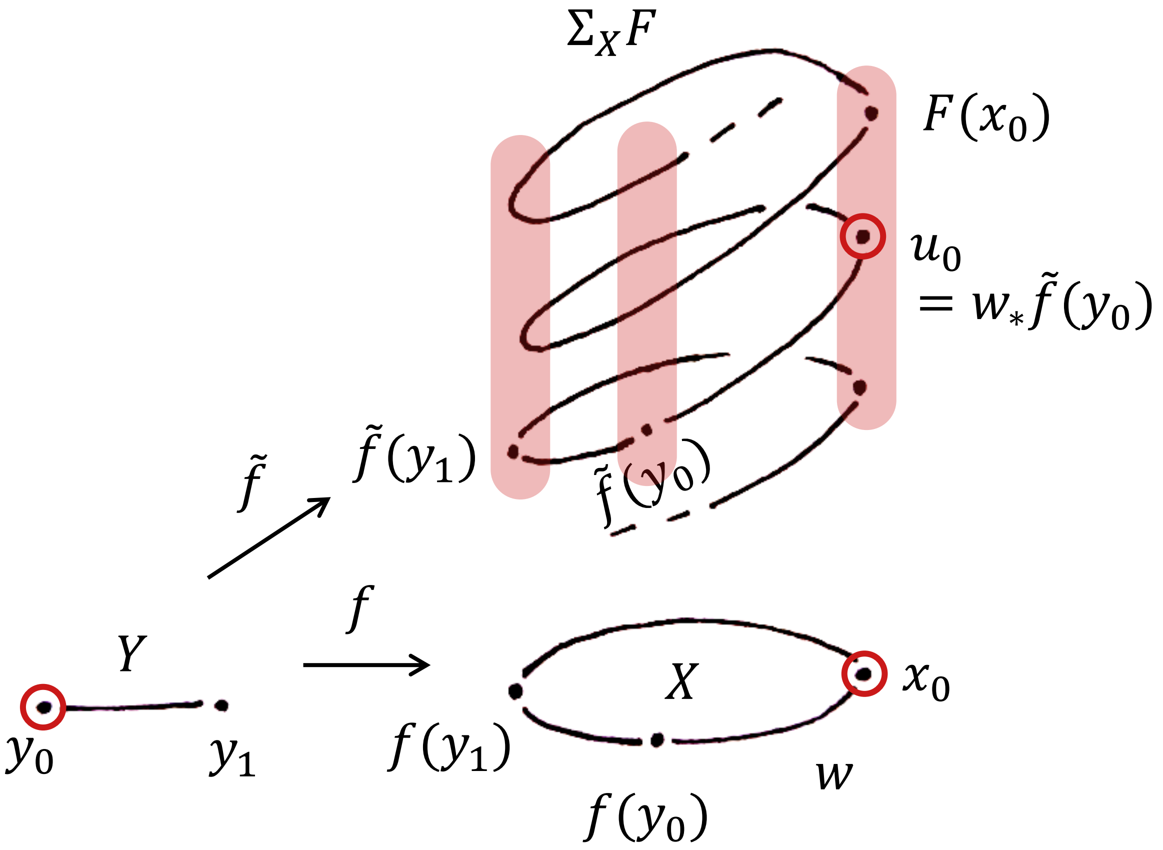

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## Lemma

*Suppose given a covering space $F : X \to \mathrm{hSet}$ with point $u : F(x_0)$ over a pointed type $(X, x_0)$ and a pointed map $f : (Y, y_0) \to (X, x_0)$ with $Y$ connected. Then a pointed lift $\tilde{f}: \prod_{y:Y} F(f(y))$ of $f$ exists iff*

$$f_*\left(\pi_{1(Y,y_0)}\right) \subset (\mathrm{pr}_1)_*\left(\pi_1\left(\Sigma_X F, (x_0; u_0)\right)\right)$$

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**Lemma (version 2)**

*Suppose given a covering space* $F : X \to \mathrm{hSet}$ *with point* $u : F(x_0)$ *over a pointed type* $(X, x_0)$ *and a pointed map* $f : (Y, y_0) \to (X, x_0)$ *with* $Y$ *connected. Then a pointed lift* $\tilde{f}\colon \prod_{y:Y} F(f(y))$ *of* $f$ *exists iff for all loops* $p : y_0 =_Y y_0$ *there exists a loop from* $u_0$ *to* $u_0$ *in* $F$ *lying over* $f_*(p)$*, i.e.*

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Proof closely reflects the **classical proof**

# Canonical Change of Basepoint

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Path from $a$ to $b$ induces a **change-of-basepoint isomorphism**

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# HoTT

Path $p : a =_X b$ also induces a change-of-basepoint isomorphism

$$\pi_n(X, a) \cong \pi_n(X, b)$$

via transport

- **Issue** $X$ **connected**, then only $\|a =_X b\|$, so only

$$\|\pi_n(X, a) \cong \pi_n(X, b)\|$$

- **Wanted** an explicit isomorphism $\pi_n(X, a) \cong \pi_n(X, b)$, considered **canonical**

Via **extension by weak constancy** [Hou (Favonia) and Harper (2016)]
it suffices to show that

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**Theorem**

*Let $X$ be a type with designated point $a : X$.*

1. *If $X$ is **simply-connected**, then the action of $\pi_1(X, a)$ on $\pi_n(X, a)$ is trivial for all $n \geq 1$*
2. *The fundamental group $\pi_1(X, a)$ is **abelian** if and only if the action on itself is trivial*
3. *If **merely** for all loops $p, q : \Omega(X, a)$, $p \cdot q = q \cdot p$ then the action of $\pi_1(X, a)$ on $\pi_n(X, a)$ is trivial for all $n \geq 1$*

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**… not always possible**

# References


- Ulrik Buchholtz, Floris van Doorn, and Egbert Rijke. **Higher Groups in Homotopy Type Theory.** In *Proceedings of the 33rd Annual ACM/IEEE Symposium on Logic in Computer Science*, LICS ’18, page 205–214, New York, NY, USA, 2018. Association for Computing Machinery.
- Kuen-Bang Hou (Favonia) and Robert Harper. **Covering Spaces in Homotopy Type Theory.** In Silvia Ghilezan, Herman Geuvers, and Jelena Ivetić, editors, *22nd International Conference on Types for Proofs and Programs (TYPES 2016)*, volume 97 of Leibniz International Proceedings in Informatics (LIPIcs), pages 11:1–11:16, Dagstuhl, Germany, 2018. Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik.