

Strictly Associative Group Theory using Univalence

Alex Rice¹

University of Cambridge

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UNIVERSITY OF
CAMBRIDGE

Outline

- 1 What did I do?
- 2 How did I do it?
- 3 Further thoughts

Motivation

```
InvUniqueLeft :  $\forall \{l\} (\mathcal{G} : \text{Group } l) \rightarrow \text{Type } l$   
InvUniqueLeft  $\mathcal{G} = \forall g h \rightarrow h \cdot g \equiv 1g \rightarrow h \equiv \text{inv } g$   
  where  
  open GroupStr ( $\mathcal{G} \text{ .snd}$ )
```

Motivation

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where

open GroupStr ($\mathcal{G} \text{ .snd}$)

$\text{inv-unique-left} : \forall \{ \ell \} (\mathcal{G} : \text{Group } \ell) \rightarrow \text{InvUniqueLeft } \mathcal{G}$

$\text{inv-unique-left } \mathcal{G} g h p =$

$h \equiv \langle \text{sym } (\cdot \text{IdR } h) \rangle$

$h \cdot 1g \equiv \langle \text{cong } (h \cdot _) (\text{sym } (\cdot \text{InvR } g)) \rangle$

$h \cdot (g \cdot \text{inv } g) \equiv \langle \cdot \text{Assoc } h g (\text{inv } g) \rangle$

$(h \cdot g) \cdot \text{inv } g \equiv \langle \text{cong } (_ \cdot \text{inv } g) p \rangle$

$1g \cdot \text{inv } g \equiv \langle \cdot \text{IdL } (\text{inv } g) \rangle$

$\text{inv } g \quad \square$

where

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inv-unique-left-strict :  $\forall \{ \ell \} (\mathcal{G} : \text{Group } \ell) \rightarrow \text{InvUniqueLeft } \mathcal{G}$   
inv-unique-left-strict  $\mathcal{G} = \text{strictify InvUniqueLeft}$ 
```

```
 $\lambda g h p \rightarrow$   
   $h \cdot 1g \quad \equiv \langle \text{cong } (h \cdot \_) (\text{sym } (\cdot \text{InvR } g)) \rangle$   
   $h \cdot g \cdot \text{inv } g \equiv \langle \text{cong } (\_ \cdot \text{inv } g) p \rangle$   
   $1g \cdot \text{inv } g \quad \square$ 
```

```
  where  
    open GroupStr (RSymGroup  $\mathcal{G} \text{ .snd}$ )  
    open import Groups.Reasoning  $\mathcal{G}$  using (strictify)
```

Strictify

- Given a group \mathcal{G} , we create a new group `RSymGroup` \mathcal{G} .

Theorem (Cayley's Theorem)

Every group is isomorphic to a subgroup of a symmetric group.

- In `RSymGroup` \mathcal{G} , various rules hold by reflexivity.
- We show that `RSymGroup` \mathcal{G} is isomorphic to \mathcal{G} .
- By univalence and the structure identity principle, `RSymGroup` \mathcal{G} is equal to \mathcal{G} .
- The `strictify` function transports a proof from `RSymGroup` \mathcal{G} back to \mathcal{G} .

In the strictified group the following equations hold definitionally:

- $a(bc) = (ab)c$,
- $a1 = a = 1a$,
- $a^{-1^{-1}} = a$,
- and $(fg)^{-1} = g^{-1} \cdot f^{-1}$.

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$$\begin{aligned} _ \circ _ &: (f : B \rightarrow C) \rightarrow (g : A \rightarrow B) \rightarrow (A \rightarrow C) \\ (f \circ g) x &= f (g x) \end{aligned}$$

$$\begin{aligned} \text{comp-assoc} &: (f : C \rightarrow D) \\ &\rightarrow (g : B \rightarrow C) \\ &\rightarrow (h : A \rightarrow B) \\ &\rightarrow f \circ (g \circ h) \equiv (f \circ g) \circ h \end{aligned}$$

$$\text{comp-assoc } f \ g \ h = \text{refl}$$

Do invertible functions compose strictly?

`record Inverse (A : Type) (B : Type) : Type where`

`field`

$\uparrow : A \rightarrow B$

$\downarrow : B \rightarrow A$

$\varepsilon : \forall x \rightarrow \downarrow (\uparrow x) \equiv x$

$\eta : \forall y \rightarrow \uparrow (\downarrow y) \equiv y$

Strict invertible functions

`record Inverse (A : Type) (B : Type) : Type where`

`constructor [_,_,_,_]`

`field`

`↑ : A → B`

`↓ : B → A`

`ε : ∀ b {x} → x ≡ ↓ b → ↑ x ≡ b`

`η : ∀ a {y} → y ≡ ↑ a → ↓ y ≡ a`

`_o_ : Inverse B C → Inverse A B → Inverse A C`

`_o_ [f , g , p , q] [f' , g' , p' , q'] =`

`[(λ x → f (f' x)) ,`
`(λ y → g' (g y)) ,`
`(λ b r → p b (p' (g b) r)) ,`
`(λ a r → q' a (q (f' a) r))]`

Strict invertible functions

$\text{assoc} : (f : \text{Inverse } C D)$
 $\rightarrow (g : \text{Inverse } B C)$
 $\rightarrow (h : \text{Inverse } A B)$
 $\rightarrow f \circ (g \circ h) \equiv (f \circ g) \circ h$

$\text{assoc } f g h = \text{refl}$

$\text{id-inv} : \text{Inverse } A A$

$\text{id-inv} = [(\lambda x \rightarrow x) , (\lambda x \rightarrow x) ,$
 $(\lambda b r \rightarrow r) , (\lambda a r \rightarrow r)]$

$\text{id-unit-left} : (f : \text{Inverse } A B)$

$\rightarrow \text{id-inv} \circ f \equiv f$

$\text{id-unit-left } f = \text{refl}$

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$\rightarrow f \circ \text{id-inv} \equiv f$

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Strict invertible functions

$\text{inv-inv} : \text{Inverse } A B \rightarrow \text{Inverse } B A$
 $\text{inv-inv } [f , g , \varepsilon , \eta] = [g , f , \eta , \varepsilon]$

$\text{inv-involution} : (f : \text{Inverse } A B)$
 $\quad \rightarrow \text{inv-inv } (\text{inv-inv } f) \equiv f$
 $\text{inv-involution } f = \text{refl}$

$\text{inv-comp} : (f : \text{Inverse } B C)$
 $\quad \rightarrow (g : \text{Inverse } A B)$
 $\quad \rightarrow \text{inv-inv } (f \circ g) \equiv \text{inv-inv } g \circ \text{inv-inv } f$
 $\text{inv-comp } f g = \text{refl}$

Representable functions

The map $\iota : g \mapsto g \cdot _$ includes the group \mathcal{G} in the symmetric group. We now want to restrict the symmetric group to those functions that are in the image of ι .

Proposition

A function $f : \mathcal{G} \rightarrow \mathcal{G}$ is in the image of ι if and only if for all $g, h \in \mathcal{G}$, $f(g \cdot h) = f(g) \cdot h$.

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Representable : Inverse $\langle \mathcal{G} \rangle \langle \mathcal{G} \rangle \rightarrow$ Type

Representable $f = \forall x g h \rightarrow x \equiv g \cdot h \rightarrow \uparrow f x \equiv \uparrow f g \cdot h$

Repr : Type

Repr = $\Sigma [f \in \text{Inverse } \langle \mathcal{G} \rangle \langle \mathcal{G} \rangle] \text{Representable } f$

Representable symmetric group

- Let $\text{RSymGroup } \mathcal{G}$ be the subgroup of the symmetric group on \mathcal{G} consisting of those functions that are representable.

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- By univalence we get an equality:

$$\iota \equiv \mathcal{G} : \mathcal{G} \equiv \mathbf{RSymGroup} \mathcal{G}$$

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- This subgroup still has strict composition.
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- By univalence we get an equality:

$$\iota \equiv \mathcal{G} : \mathcal{G} \equiv \text{RSymGroup } \mathcal{G}$$

- This lets us define:

```
strictify : (G : Group ℓ-zero)
           → (P : Group ℓ-zero → Type)
           → P (RSymGroup G)
           → P G
```

```
strictify G P p = transport (sym (cong P (ι ≡ G))) p
```

Further thoughts

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Does this all work with categories instead of groups?

Conclusion

- For each group \mathcal{G} we can generate an isomorphic group `RSymGroup` \mathcal{G} .
- This group has nice definitional properties
- Univalence allows us to generate an equality between the two groups.
- This allows us to prove theorems about an arbitrary group by instead proving them on the strictified group.
- <https://alexarice.github.io/posts/sgtuf/Strict-Group-Theory-UF.html>