HoTT/UF 2023 Talk

21 April 2023 16:53

∞-Topoi as Internal ∞-Categories

Manifesto for internal ∞-categories!!!

Synthetic Algebraic Topology

Result	Classical	HoTT
Fundamental group of circle	Brouwer Hopf 20'	Licata-Shulman '13
Blakers-Massey	'49	HoTT Book '13 Agda '16
Computing $\pi_4(S^3)$	Freudenthal '37	Brunerie '16
Truncation	Postnikov 51, '60	Rijke '17 RSS '20 CORS '20
Serre Finiteness	Serre '53	? ⇐
Stabilization	Lima 58 Adams 60-70	van Doorn 18, ?
Chromatic Homotopy	Adams 63	?
Homotopy Algebraic Structures	Operads May 72 Lawvere thy 63	?

Sequential Colimits: Rijke '17
SvDR '20
R. 21

Localizations: { RSS
Vergura
ABFJ

Doesn't always work: presentability, adjoint functor theorem, ..

Conjecture/ Let $\mathcal{E}$ be a locally Cartesian closed ∞-cat with a universe $\mathcal{U}$ (TT w/ Π-types, Σ-types, univalent universe, path types, ...).

- There exists a notion of internal ∞-cat in $\mathcal{E}$, ✓ and it comes with a cat thy. ✗
- $\mathcal{E}$ can be internalized to an $\mathcal{E}$-∞-cat $\underline{\mathcal{E}}$ s.t. $\mathrm{Map}(1, \underline{\mathcal{E}}) \simeq \mathcal{E}^{small}$ (up to size issues) ✗
- $\underline{\mathcal{E}}$ is cocomplete (internally) and in fact the cocompletion of the terminal $\mathcal{E}$-∞-cat. ✗ (if we assume geom. real.)

Implications

- $\mathbb{N}$ NNO $\rightsquigarrow$ $\underline{\mathcal{E}}^{\mathbb{N}} \xrightarrow{colim} \underline{\mathcal{E}}$.
- $\tau_n \underline{\mathcal{E}}$ is cocomplete $\rightsquigarrow$ $\tau_n : \underline{\mathcal{E}} \to \tau_n \underline{\mathcal{E}}$ the unique functor that preserves the terminal obj.

- $\underline{A}$ $\mathcal{E}$-∞-cat of algebras ⤳

$$\begin{tikzcd}
\underline{\mathcal{E}} \arrow[r, "\text{Free}"] & \underline{A} \\
1 \arrow[u] \arrow[ur, dashed] &
\end{tikzcd}$$

Cat Thy vs. Type Thy internally.

TT	CT
Types	Obj
Terms	Mor
Constructors	Universal Prop

		TT-like	CT-like
Cat Thy	Subobject Classifier	Ω, $\mathrm{Sub}(X) \cong \mathrm{Hom}(X,\Omega)$ $\wedge : \Omega \times \Omega \to \Omega$ ⟶	Ω Heyting alg
∞-Cat Thy	Object Classifier	$\mathcal{U}$, $(\mathcal{E}_{/X})^{\simeq} \simeq \mathrm{Map}(X,\mathcal{U})$ ⤳	$\mathcal{U}$ should have ∞-cat structure?
TT	$\mathcal{U}$		?

Constructing Internal ∞-Categories

Def/ Let $\mathcal{E}$ be as before. An internal ∞-cat is a simplicial obj $W: \Delta^{op} \to \mathcal{E}$ that satisfies some finite limit conditions.

- complete Segal obj
- Rezk obj

- Rezk type

Thm/ (R.) Let $\mathcal{E}$ be as before. Then there exists an internal ∞-cat $\underline{\mathcal{E}} : \Delta^{op} \to \mathcal{E}$ s.t.

- $\underline{\mathcal{E}}_0 \simeq \mathcal{U}$
- $\mathrm{Map}(X, \underline{\mathcal{E}}) \simeq (\mathcal{E}_{/X})^{small}$

Is $\underline{\mathcal{E}}$ cocomplete?

A small detour: The cocomplete Case:

Let us also assume that $\mathcal{E}$ has geometric realization.

$$\mathrm{colim} : \mathcal{E}^{\Delta^{op}} \longrightarrow \mathcal{E}.$$

Theorem (-ish)/(R) If $\mathcal{E}$ is as above, then $\underline{\mathcal{E}}$ is the cocompletion of the point.

Proof Sketch/ Martini
Wolf

- $\underline{\mathcal{E}}$ has a terminal obj $\underline{\mathcal{E}}_* \longrightarrow \underline{\mathcal{E}}$, which is the universal left fibration.

$$\mathrm{Fun}(\underline{\mathcal{C}}, \underline{\mathcal{E}}) \simeq \mathcal{L}Fib_{/\underline{\mathcal{C}}}$$

- $s\mathcal{U} \to \underline{\mathcal{E}}$ $\quad \mathrm{Map}(X, s\mathcal{U}) \to \mathrm{Map}(X, \underline{\mathcal{E}})$
$s\mathcal{E}_{/X} \to \mathcal{E}_{/X}$

$$\begin{tikzcd}
\mathrm{Fun}(\underline{\mathcal{C}}, \underline{\mathcal{E}}) \arrow[r, phantom, "\simeq"] \arrow[rrrr, bend right] & \mathcal{L}Fib_{/\underline{\mathcal{C}}} \arrow[r, hook] & s\underline{\mathcal{E}}_{/\underline{\mathcal{C}}} \arrow[r] & s\underline{\mathcal{E}} \arrow[r, "\mathrm{colim}"] & \underline{\mathcal{E}}
\end{tikzcd}$$

Back to the Road: The general Case.

Having geometric realization is necessary.

Ex/ Take the filter quotient ∞-topos with the Frechet filter on $\mathbb{N}$.

$$\prod_F \mathcal{S} : \begin{cases} \text{Obj} : & (X_n)_{n\in\mathbb{N}} \quad X_n \text{ a top. space} \\ \text{Mor} : & f_n : X_n \to Y_n \text{ for } n > N \\ & \text{up to eventual equality.} \end{cases}$$

$$\prod_{\mathbb{N}} \mathcal{S} \longrightarrow \prod_F \mathcal{S}$$

preserves finite limits & colimits, universes, subobj classifier, CCC

does not preserve presentability or countable colimits.

$$(S^0)^{\bullet} \Delta^{op} \longrightarrow \prod_F \mathcal{S} \quad \text{as model structure}$$

$$n \longmapsto (S^0)^{\times n}$$

$$\operatorname{colim}_{\Delta^{op}} (S^0)^{\bullet} = \operatorname{colim}_{\Delta^{op}_{\le 0}} (S^0)^{\bullet} \to \operatorname{colim}_{\Delta^{op}_{\le 1}} (S^0)^{\bullet} \to \cdots$$

$$= \operatorname{colim}\left(\underline{S^0 \longrightarrow S^1 \to S^2 \to \cdots \cdots \overset{\checkmark}{\to} S^\infty}\right)$$

Notice

$$\begin{tikzcd} \prod_J \mathrm{Top} \arrow[r, "U"] & \prod_J \mathrm{Set} \arrow[l, bend right, "\mathrm{Disc}"'] \arrow[l, bend left, "\mathrm{Codisc}"] \end{tikzcd}$$

$$\operatorname{colim}(US^n) = \coprod US^n \setminus US^{n-1} \quad \text{which doesn't exist.}$$

Where does this leave us?

In a tricky place.

What we should do is Δ_Σ as an Σ-∞-cat.

Then define truly internal Σ-∞-cat as a

$\Delta_\Sigma \longrightarrow \underline{\Sigma}$. that satisfies the right conditions.