Higher Pro-arrows: Towards a Model for Naturality Pretype Theory

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Vienna, Austria
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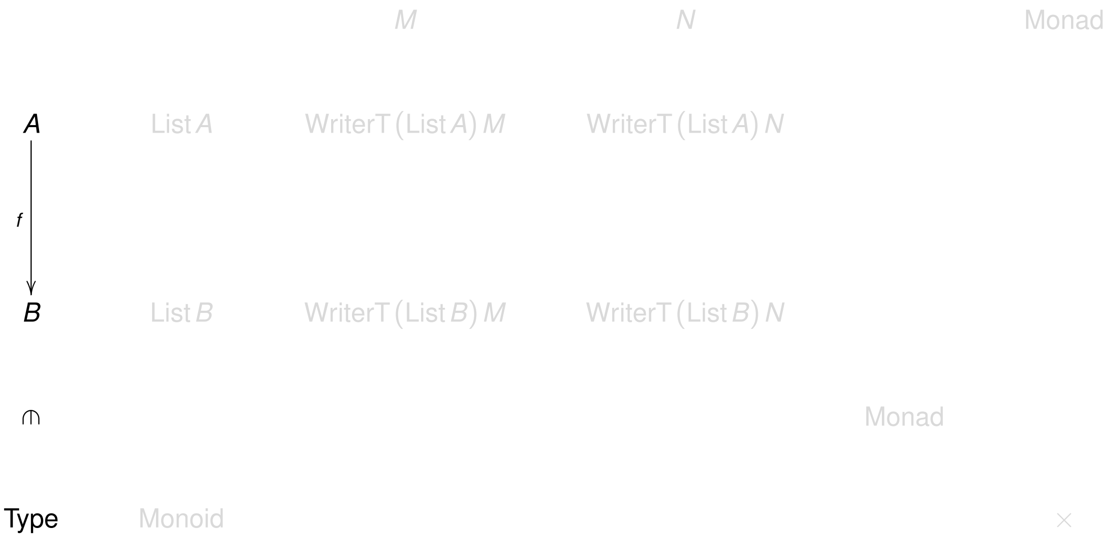
Naturality TT: **Why?** (And what?)
Example problem in verified functional programming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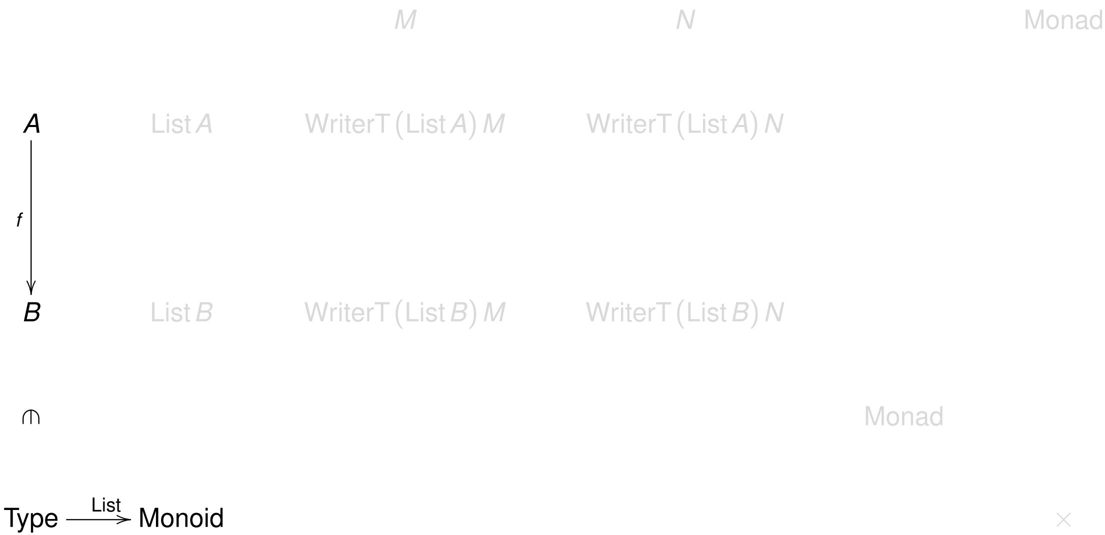
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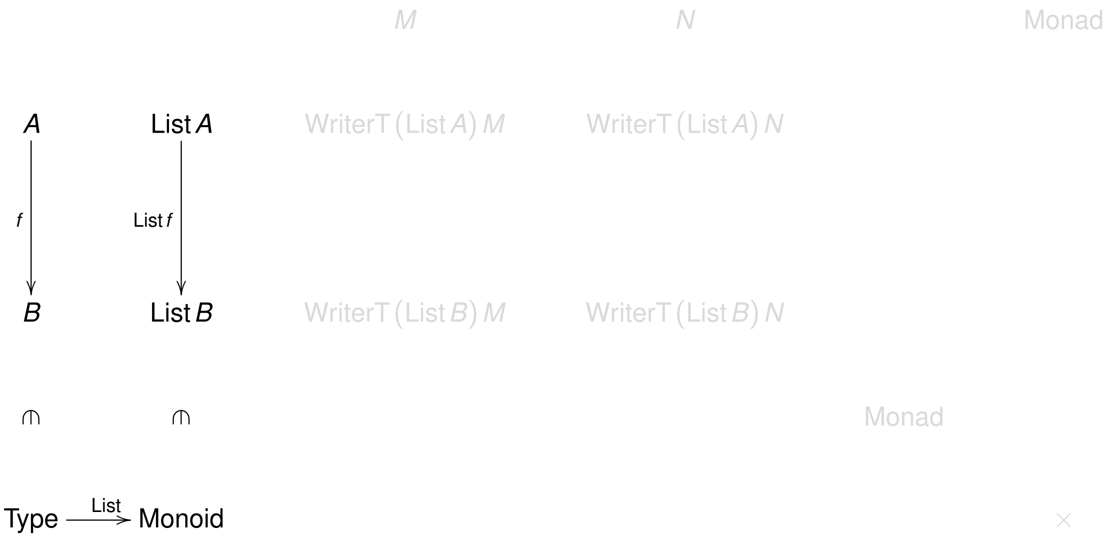
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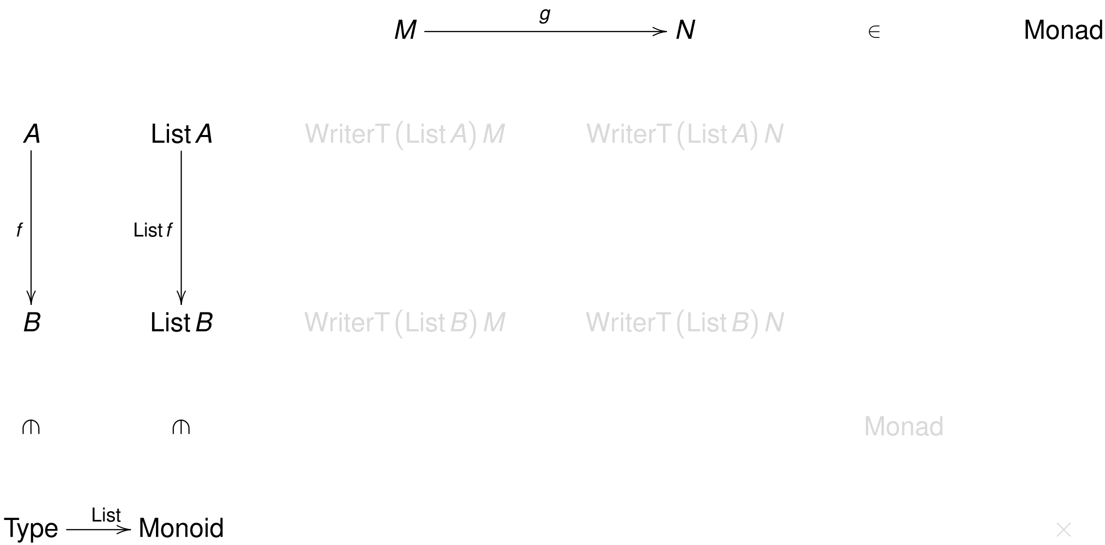
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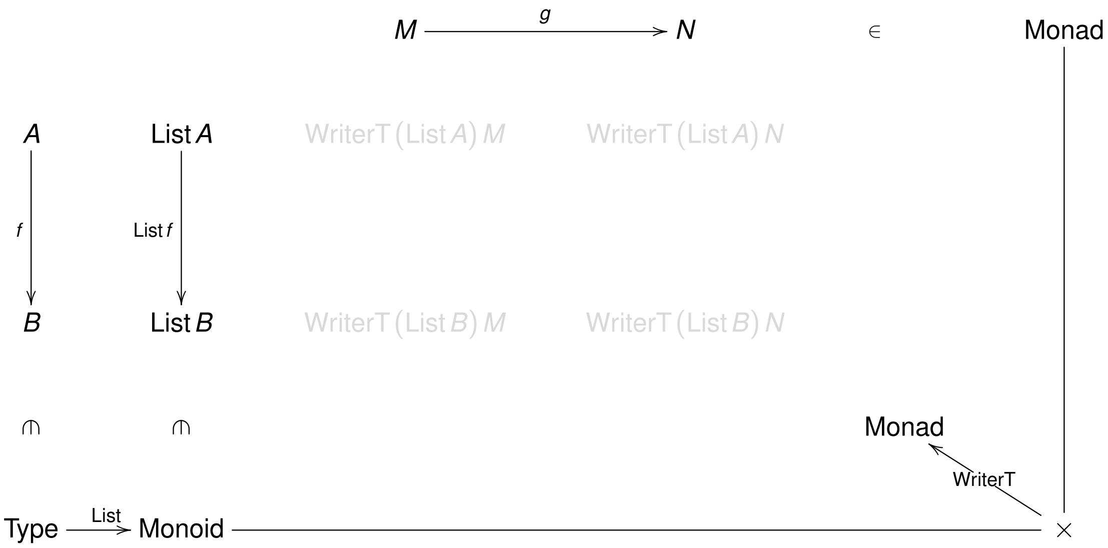
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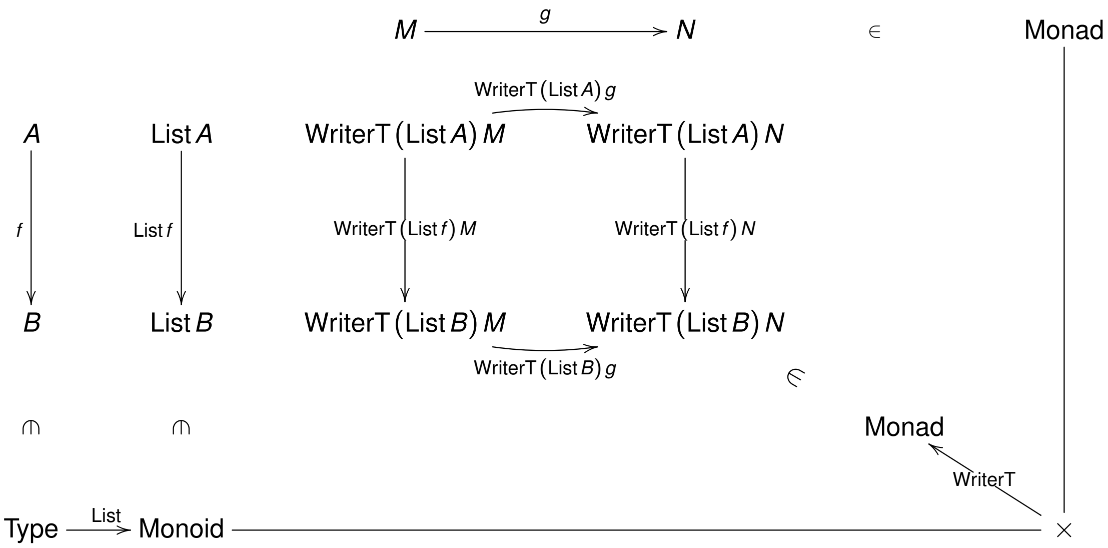
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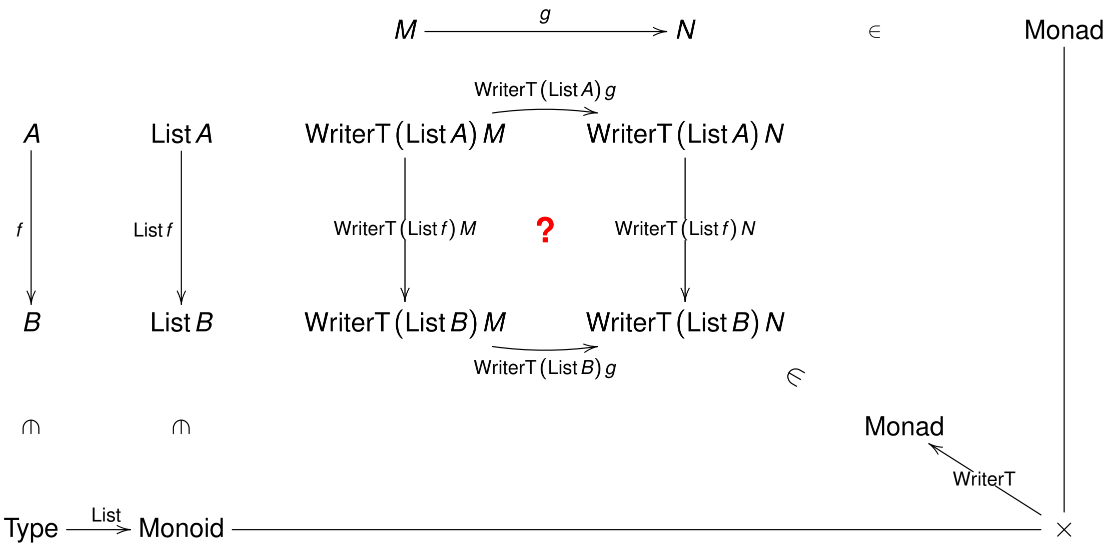
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In plain DTT

Functoriality of List : Type $\to$ Monoid:

- Object action: $(\text{List}\, A, [], ++)$
- Functorial action:
  - List f : List $A \to$ List B **(by recursion)**
  - List f is a monoid morphism:
    - List f preserves [] (trivial)
    - List f preserves $++$ **(by induction)**

  + functor laws **(by induction)**

Functoriality of
WriterT : Monoid $\to$ MonadTrans

- Object action: WriterT $W \in$ MonadTrans
  - Object action: WriterT $W\, M \in$ Monad
    - Object action: Define WriterT $W\, M\, A$
    - Functorial action WriterT $W\, M\, f$
      + functor laws
    - return & bind + naturality

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    + functor laws
  - lift : $M \to$ WriterT $W\, M$ + naturality
    - Respects return & bind
- Functorial action:
  WriterT h : WriterT $V \to$ WriterT W
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In parametric DTT

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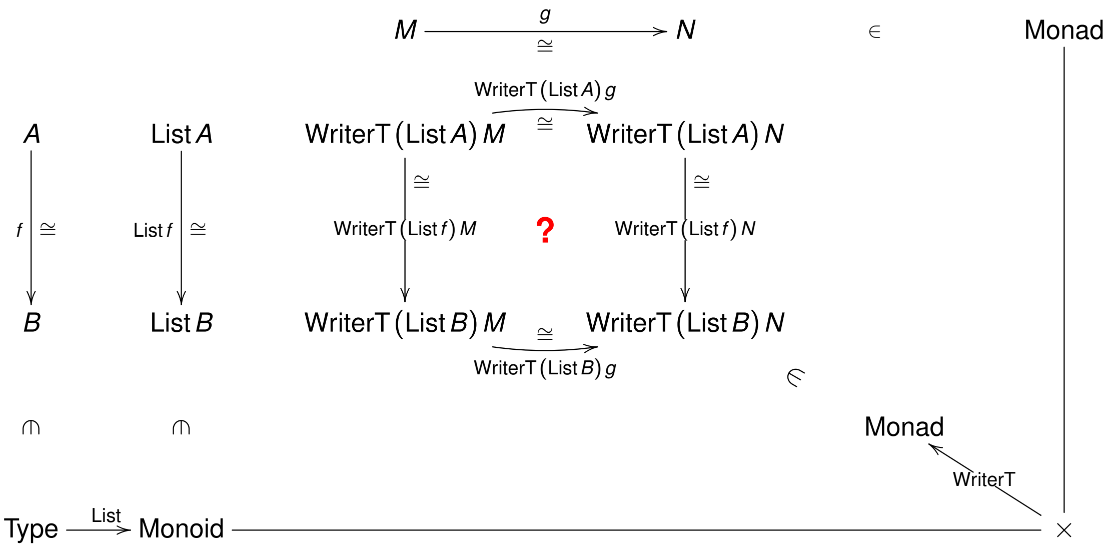

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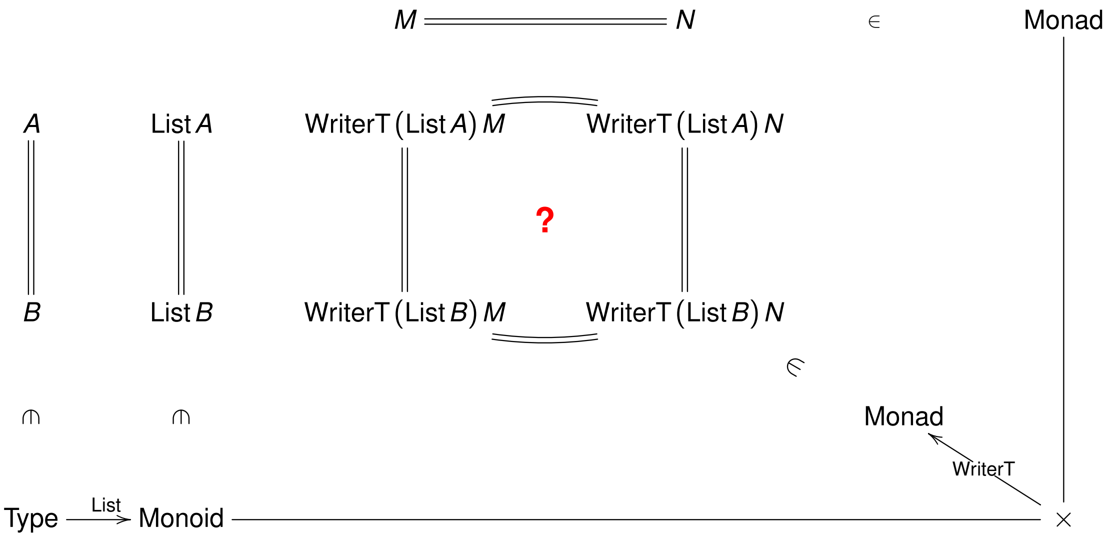
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In HoTT (assuming f, g and $h = \text{List}\, f$ are isos)

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In Naturality TT

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Variance and modalities

WriterT $W\,M\,A$: Monad is **covariant** w.r.t.

- W : Monoid
- M : Monad
- A : Type

ReaderT $R\,M\,A$ is **contravariant** w.r.t.

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return : $A \to$ WriterT $W\,M\,A$ is **natural** w.r.t.

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Ignoring variance

- HoTT: only consider **isomorphisms**
  ☹ Not everything is an isomorphism.
- Param'ty: **relations**, not morphisms
  ☹ Don't know how to compute fmap.

Naturality TT

- Preserve isomorphisms
- Preserve relations
- Keep track of action on morphisms

Hence:

- Use functoriality/naturality when possible
- Use HoTT when applicable
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Pretypes: A Note on Fibrancy

A presheaf model of DTT can account for the **existence** of **paths/morphisms/bridges/...**

Fibrant types have **operations** for these:

Directed	
functorial	Transport along morphisms
Segal	Composition of morphisms
Rezk	Isomorphism-path univalence

HoTT	
Kan	Comp. of & transp. along paths

Param'ty	
discrete	Homog. bridges express equality
...	...

We **ignore** fibrancy for now:

- Functoriality & Segal fibrancy are brittle
  $\Rightarrow$ need to consider pretypes anyway
- There are promising techniques for defining fibrancy internally:
  - Contextual fibrancy [BT21, Nuy20]
  - Amazing right adjoint [LOPS18] & Transpension [ND21]
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Model-first Approach

The type system emerges from the model:

- A diagram of CwFs and adjunctions models an instance of MTT [GKNB20].
- An endofunctor on $\mathscr{W}$ models a substructural shape (e.g. $\mathbb{I}$) in $\mathrm{Psh}(\mathscr{W})$ giving rise to modalities $\exists(i : \mathbb{I}) \dashv ⅎ[i : \mathbb{I}] \dashv \forall(i : \mathbb{I}) \dashv \between[i : \mathbb{I}]$. This is the basis of the modal transpension type system (MTraS) [ND21].

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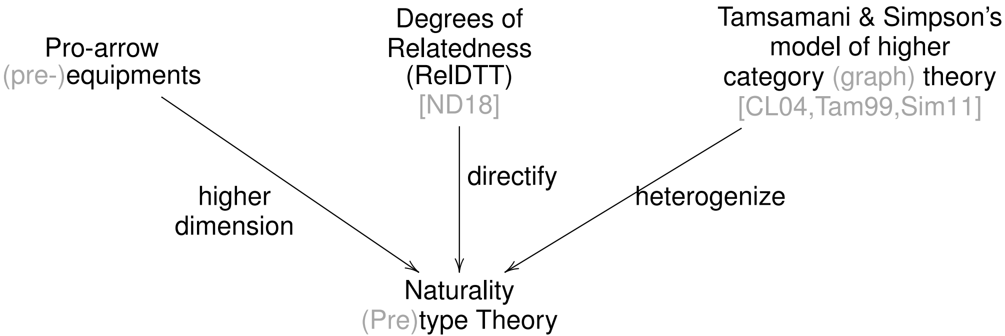
The Model


Pro-arrow
(pre-)equipments
Degrees of
Relatedness
(RelDTT)
[ND18]
Tamsamani & Simpson's
model of higher
category (graph) theory
[CL04,Tam99,Sim11]
higher
dimension
directify
heterogenize
Naturality
(Pre)type Theory

n-Fold Categories

Category

A **category** $\mathscr{C}$ can be defined as a **simplicial set** $\mathscr{C} \in \mathrm{Psh}(\Delta)$ satisfying the **Segal condition**.

n-Fold category

An **n-fold category** $\mathscr{C}$ is an **n-fold simplicial set** $\mathscr{C} \in \mathrm{Psh}(\Delta^n)$ satisfying the **Segal condition** in each dimension.

Double category

A **double category** $\mathscr{C}$ has:

- objects
- horiz. arrows / (1)-arrows
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Pretypes!

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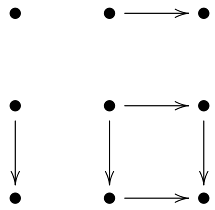

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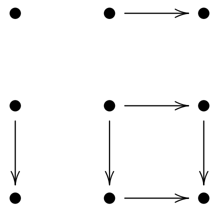

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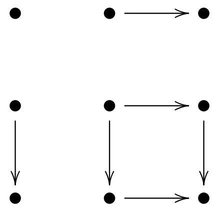

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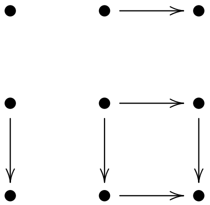

Pretypes!

The Twisted Prism Functor

Δ is a skeleton of FinLinOrd,
hence $\Delta \simeq$ FinLinOrd.

Twisted Prism Functor [PK19]

$\sqcup \ltimes \mathbb{I}$: FinLinOrd $\to$ FinLinOrd :
$W \mapsto W^{\mathrm{op}} \uplus_{<} W$

$$a \longrightarrow b \quad \mapsto \quad \begin{array}{ccc} \iota_0\, a & \longleftarrow & \iota_0\, b \\ \downarrow & & \downarrow \\ \iota_1\, a & \longrightarrow & \iota_1\, b \end{array}$$

An MTraS-shape $\mathbb{I}$ modelled by $\sqcup \ltimes \mathbb{I}$, reconciles the view of Hom as a **contra-/covariant bifunctor** with a view as a directed **path type**.

$\mathbb{I}$ as an MTraS-shape is better behaved on $\bowtie$:

Twisted Cube Category $\bowtie$ [PK19]

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(Pro-arrow) Equipments

(Pro-arrow) Equipment

An **equipment** $\mathscr{C}$ is a **double category** with

- objects
- arrows ($\to$)
- pro-arrows ($\not\to$)
- squares

such that every arrow $\varphi : x \to y$ has **graph** pro-arrows $\varphi^{\ddagger} : x \not\to y$ and $\varphi^{\dagger} : y \not\to x$ such that $(\ldots)$.

Example: Set

Set is an equipment with:

- sets
- functions
- relations
  - identity relation: equality
  - $(R; S)(x, z) = \exists y. R(x, y) \wedge S(y, z)$
- proofs that $R(a, b) \Rightarrow S(f\,a, g\,b)$

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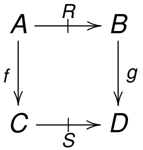

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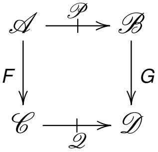
Example: Cat

Cat is an equipment with:

- categories
- functors
- profunctors
  - identity profunctor: Hom
  - $(\mathscr{P};\mathscr{Q})(x,z) = \overset{\text{coend}}{\exists}\, y.\mathscr{P}(x,y) \times \mathscr{Q}(y,z)$
- $\overset{\text{end}}{\forall}\, a,b.\mathscr{P}(a,b) \Rightarrow \mathscr{Q}(F\,a, G\,b)$



Higher equipments

Set is . . .

- A large set
- A category
- An equipment

Cat is . . .

- A category
- A 2-category
- An equipment

Eqmnt is . . .

- An equipment
- A 2-equipment

Eqmnt has:

Objects	Equipments
Arrows	Equipment functors
Pro-arrows	Equipment profunctors: Contain arrows and pro-arrows
Pro-pro-arrows	Equipment **pro-profunctors:** Contain pro-arrows
Squares	. . .
Cubes	. . .

Higher Equipment

An *n*-**equipment** is an *n*-**fold category** (. . .)

$\Rightarrow \mathscr{C} \in \mathrm{Psh}(\bowtie^n_{\dagger,\ddagger})$

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Directifying “Degrees of Relatedness” [ND18]

Depth n types

- i-edge relations $\frown_i$
- $R : A \frown^{\mathrm{U}}_{i+1} B$
  is a container for $r : a \frown^{R}_{i} b$
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- $a \frown_i b \Rightarrow a \frown_{i+1} b$
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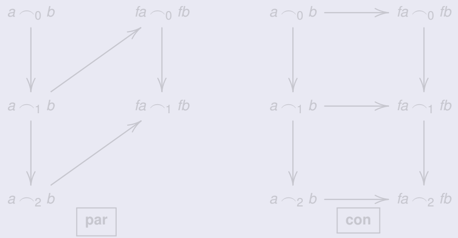

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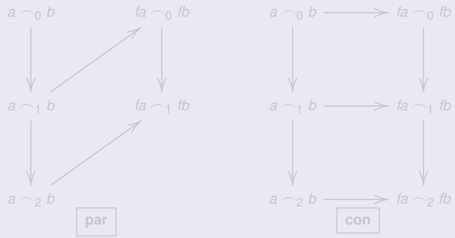

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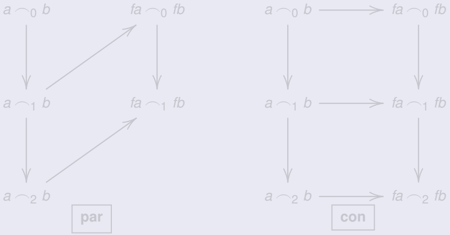

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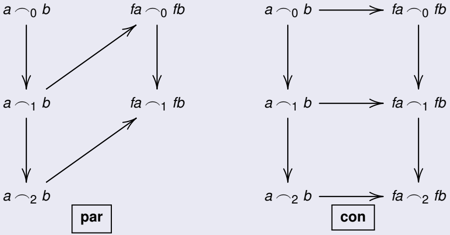

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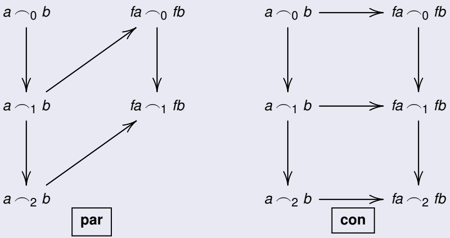

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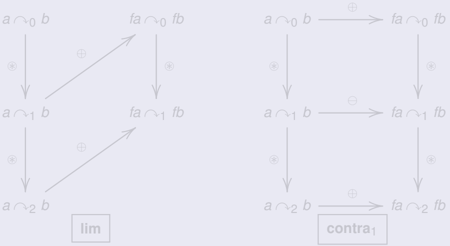

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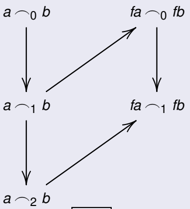

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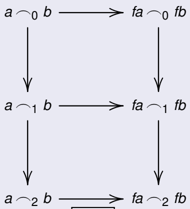

con

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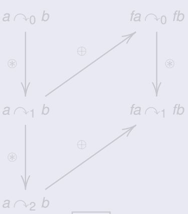

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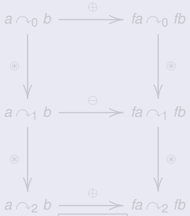

contra$_1$

Directifying "Degrees of Relatedness" [ND18]

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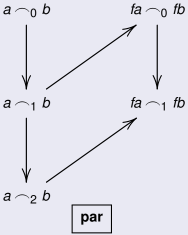

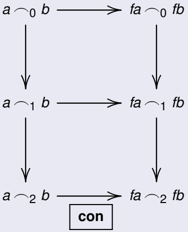

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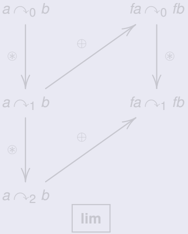

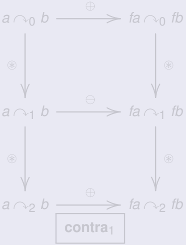

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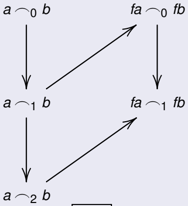

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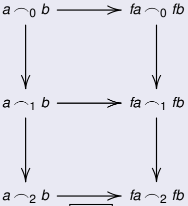

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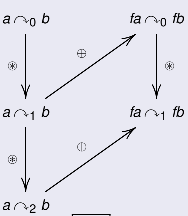

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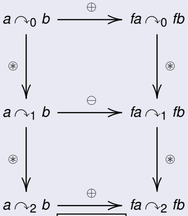

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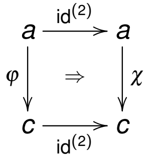

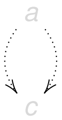

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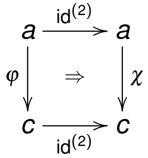

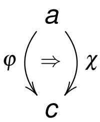

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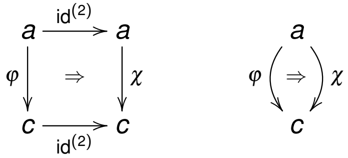

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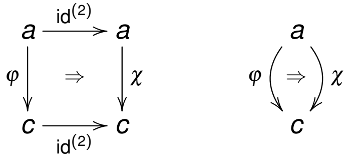

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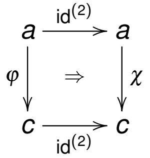

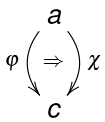

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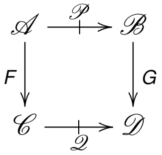

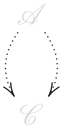

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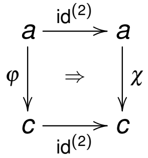

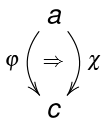

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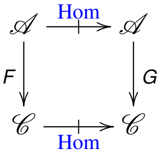

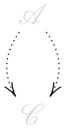

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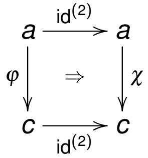

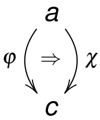

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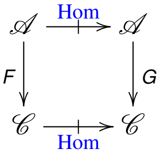

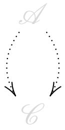

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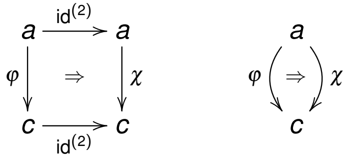

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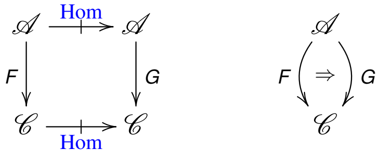

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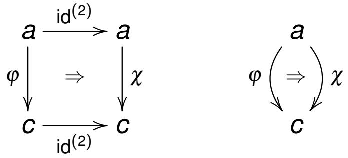

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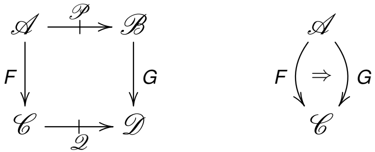

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