The Category Interpretation of Polarized and Directed Type Theory

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Where to find more detail

- These slides:
  jacobneu.github.io/research/slides/HoTT-UF-2023.pdf
- A preprint will appear here:
  jacobneu.github.io/research/preprints/polarTT.pdf
- Agda formalization coming soon (link will be added to preprint and slides)

Univalent Mathematics:

Groupoid Theory

versus

Category Theory

$\infty-$groupoids are easy in HoTT

Recall A type in HoTT can be viewed as a ∞-**groupoid**: the elements are the objects, the identity proofs are the morphisms, ...

A function $f : A \to B$ is automatically a functor w.r.t. this groupoid structure: using the J-rule, we can construct $\mathrm{ap}_f\ p \colon f(a) =_B f(a')$ for each $p : a =_A a'$ and prove this preserves identities (refl) and composition (path concatenation)

Key observation We don't need to inspect the definition of f to define ap_f or to prove it respects identities and composition – once we have f, we have its functoriality

Not so for univalent category theory

To define a category, we must define its morphisms explicitly and prove they satisfy the given laws.

To define a functor, we must define its morphism part explicitly and prove functoriality by hand.

If we want to do ∞-category theory...

Summary: In univalent mathematics, groupoids are *synthetic* but categories are *analytic*

Why?

Moral: We *ought* to do directed type theory

Some Existing Directed TT/Synthetic CT Projects

- Harper and Licata – *2-Dimensional Directed Type Theory* (2011) †⋆
- Nuyts – *Towards a Directed Homotopy Type Theory based on 4 Kinds of Variance* (2015) †⋆
- Riehl and Shulman – *A type theory for synthetic ∞-categories* (2017)
- Ahrens, North, and van der Wiede – *Semantics for two-dimensional type theory* (2022) ⋆
- Cisinski, Nguyen, and Walde – *Univalent Directed Type Theory* (2023)

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† No general definition of model, though perhaps includes semantics/interpretation

⋆ *2-dimensional* – encodes morphisms between substitutions

Contribution

Directed TT using CwFs

Deep Polarity*

$\star$ Harper & Licata's work does partially address deep polarity

Back in the 90s. . .

Our goal is to develop directed type theory following in the tradition of several landmark papers from the 1990s that paved the way for homotopy type theory:

- Dybjer's *Internal Type Theory* (1995)
  - Introduced **categories with families** as a model theory for type theory
  - Generalized algebraic theory – more convenient to formalize in a computer proof assistant
- Hofmann and Streicher's *The Groupoid Interpretation of Type Theory* (1995)
  - Introduced the **groupoid model** of type theory, a CwF structure on the category of groupoids
  - Proved the independence of the *Uniqueness of Identity Proofs*

Next up in the *Directed CwFs Transatlantic Tour*

At the HoTT Conference (May 2023, Pittsburgh, USA), we'll present presheaf semantics for directed type theory, a directed analogue of these works:

- Hofmann and Streicher's *Lifting Grothendieck Universes* (1999, unpublished)
  - Established a technique for modelling universes in *presheaf models* of type theory
- Hofmann's *Semantical analysis of higher-order abstract syntax* (1999)
  - Gave presheaf semantics for a **higher-order abstract syntax**, which abstracts away cumbersome details about substitution and binding

Defn. A **category with families (CwF)** is a (generalized) algebraic structure, consisting of:

- A category Con of *contexts* and *substitutions*, with a terminal object $\bullet$, the *empty context*
- A presheaf $\mathsf{Ty} \colon \mathsf{Con}^{\mathsf{op}} \to \mathsf{Set}$ of *types*
- A presheaf $\mathsf{Tm} \colon (\int \mathsf{Ty})^{\mathsf{op}} \to \mathsf{Set}$ of *terms*
- An operation of *context extension*:

$$\frac{\Gamma \colon \mathsf{Con} \qquad A \colon \mathsf{Ty}\,\Gamma}{\Gamma \rhd A \colon \mathsf{Con}}$$

so that $\Gamma \rhd A$ is a 'locally representing object' (in the sense spelled out on the next slide)

The Local Representability Condition

For any Δ, Γ and any $A : \mathsf{Ty}\,\Gamma$,

$$\mathsf{Con}(\Delta, \Gamma \triangleright A) \quad \cong \quad \sum_{\gamma\,:\,\mathsf{Con}(\Delta,\Gamma)} \mathsf{Tm}(\Delta, A[\gamma])$$

natural in Δ.

The **groupoid model of type theory** is a CwF where

- Con is the category of groupoids
- Ty Γ is the set of Γ-indexed families of groupoids (i.e. functors $\Gamma \to \text{Grpd}$)
- $\ldots$

Further structure Can interpret dependent types and identity types in the groupoid model, and find types whose identity types violate UIP

Main Idea: Replace groupoids with categories!

The category model of type theory is a CwF where

- Con is the category of **categories**
- Ty Γ is the set of Γ-indexed families of **categories** (i.e. functors $\Gamma \to \mathsf{Cat}$)
- ...

Further structure The category of categories comes equipped with the **opposite category** operation, which we can view as a functor $\mathsf{Cat} \to \mathsf{Cat}$.

- For each context Γ, there is a context Γ^-
- For each $A : \mathsf{Ty}\ \Gamma$, there is a type $A^- : \mathsf{Ty}\ \Gamma$

A **polarized category with families (PCwF)** is a (generalized) algebraic structure, consisting of:

- Con, $\bullet$, Ty, Tm as in the definition of CwF
- A functor $(\_)^-$: Con $\to$ Con such that $(\Gamma^-)^- = \Gamma$ and $\bullet^- = \bullet$
- For each Γ: Con, a function $(\_)^-$: Ty $\Gamma \to$ Ty Γ such that $(A^-)^- = A$
- Two operations of *context extension*: for s either $+$ or $-$,

$$\frac{\Gamma : \mathrm{Con} \quad A : \mathrm{Ty}\ \Gamma^s}{\Gamma \triangleright^s A : \mathrm{Con}}$$

The Local Representability Conditions

For any Δ, Γ and any $A \colon \mathsf{Ty}\,\Gamma$,

$$\mathsf{Con}(\Delta, \Gamma \rhd^s A) \quad \cong \quad \sum_{\gamma \colon \mathsf{Con}(\Delta,\Gamma)} \mathsf{Tm}(\Delta^s, A[\gamma^s]^s)$$

natural in Δ.

Further structure In the groupoid model, we were able to interpret *identity types*. In the category model, we have **hom types**.

$$\frac{A : \mathrm{Ty}\,\Gamma \quad a_0 : \mathrm{Tm}(\Gamma, A^-) \quad a_1 : \mathrm{Tm}(\Gamma, A)}{a_0 \Rightarrow_A a_1 : \mathrm{Ty}\,\Gamma}$$

Note the use of polarities to mark variances!

Notice This is the essential ingredient in making our types into **synthetic categories**.

Further structure The groupoid model also 'lives inside' the category model: we can take the **core** of a category $\mathbb{C}$, which is the largest groupoid that is a subcategory of $\mathbb{C}$ (and of $\mathbb{C}^{\mathrm{op}}$). We could perhaps treat this as an operation on contexts, but we're mainly interested in it at the type level:

$$\frac{A : \mathrm{Ty}\,\Gamma}{A^0 : \mathrm{Ty}\,\Gamma} \qquad \frac{a : \mathrm{Tm}(\Gamma, A^0)}{+a : \mathrm{Tm}(\Gamma, A) \qquad -a : \mathrm{Tm}(\Gamma, A^-)}$$

Core types allow us to state the **introduction rule** for hom types:

$$\frac{a \colon \mathrm{Tm}(\Gamma, A^0)}{\mathrm{refl}_a \colon \mathrm{Tm}(\Gamma, -a \Rightarrow_A +a)}$$

as well as the appropriate **J-rules**: for any $a' : \mathrm{Tm}(\Gamma, A^0)$

$$\frac{m \colon \mathrm{Tm}(\Gamma, M(+a', \mathrm{refl}_{a'})) \quad a'' \colon \mathrm{Tm}(\Gamma, A) \quad q \colon \mathrm{Tm}(\Gamma, -a' \Rightarrow a'')}{J^+_M\ m\ q \colon \mathrm{Tm}(\Gamma, M(a'', q))}$$

$$\frac{n \colon \mathrm{Tm}(\Gamma, N(-a', \mathrm{refl}_{a'})) \quad a \colon \mathrm{Tm}(\Gamma, A^-) \quad p \colon \mathrm{Tm}(\Gamma, a \Rightarrow +a')}{J^-_N\ n\ p \colon \mathrm{Tm}(\Gamma, N(a, p))}$$

Proof of concept: Composition

Given

- $x : \mathrm{Tm}(\Gamma, A^-)$
- $y : \mathrm{Tm}(\Gamma, A^0)$
- $z : \mathrm{Tm}(\Gamma, A)$
- $f : \mathrm{Tm}(\Gamma, x \Rightarrow +y)$
- $g : \mathrm{Tm}(\Gamma, -y \Rightarrow z)$

Define $f \cdot g : \mathrm{Tm}(\Gamma, x \Rightarrow z)$ as either

$$J^+_M\ f\ g \qquad \text{or} \qquad J^-_N\ g\ f$$

where

$$M(a'', q) :\equiv x \Rightarrow a'' \qquad \text{and} \qquad N(a, p) :\equiv a \Rightarrow z$$

Directed Categories with Families

A **directed category with families (DCwF)** is a (generalized) algebraic structure, consisting of:

- Con, $\bullet$, Ty, Tm as in the definition of CwF
- The negation operations $(\_)^-$ and context extensions $\triangleright^s$ as in the definition of PCwF
- Core types and the $+$ and $-$ operations on terms
- The $\_ \Rightarrow \_$ type former with refl constructor and J eliminators

Thank you!