Does Path Induction Need a Justification?

A Husserlian Philosophy of HoTT

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Background: Justifying Path Induction

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- P. Walsh (2017): Yes, path induction (identity elimination rule) can be justified to be **harmonious to the identity introduction rule**
- Me: does path induction actually need a justification?

Method: Husserlian Philosophy of HoTT

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- Enter the 'community of empathy' with the mathematicians
- Look at the mathematics from the perspective of the mathematical community, while understanding what motivated the mathematicians.

Goals of the Talk

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- Explain the perspectives of the HoTT-community
- Prove that path induction follows from the path lifting property
- Thus: path induction does not need a pre-mathematical/harmonious justification.

Structure

1. HoTT community's perspective: Univalence
2. Path Induction
3. Path Lifting to Path Induction

Outline

HoTT according to the HoTT-community

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  [...] a new conception of foundations of mathematics, with **intrinsic homotopical content**,*[...] and* **convenient machine implementations**, *which can serve as a practical aid to the working mathematician. (Univalent Foundations Program, 2013, Introduction)*

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(3) homotopical interpretation: a **homotopy equivalence** p between topological spaces A and B.

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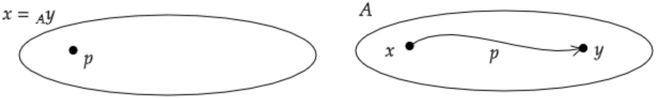

Outline

Path Induction

Definition (Path induction)

Let A be a type, and

$$C : \prod_{x,y:A} (x =_A y) \to \mathcal{U}$$

be a family of types, and let a function

$$c : \prod_{x:A} C(x, x, \mathsf{refl}_x)$$

be such that $c(x) : C(x, x, \mathsf{refl}_x)$.

Path Induction

Definition (Path induction (Cont'd))

Then there is a function

$$f : \prod_{x,y:A} \prod_{p:x=y} C(x,y,p),$$

such that

$$f(x,x,\mathsf{refl}_x) \equiv c(x).$$

PI explained (via UA)

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So given $c(x) : C(x, x, \mathsf{refl}_x)$, path induction gives us a function f that generalises it to $f(x, y, p) : C(x, y, p)$.

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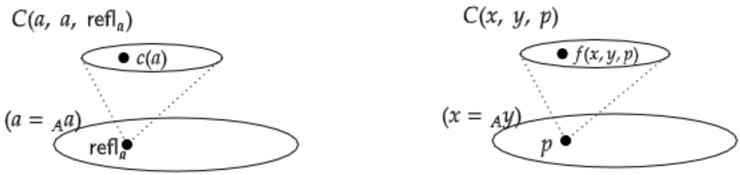

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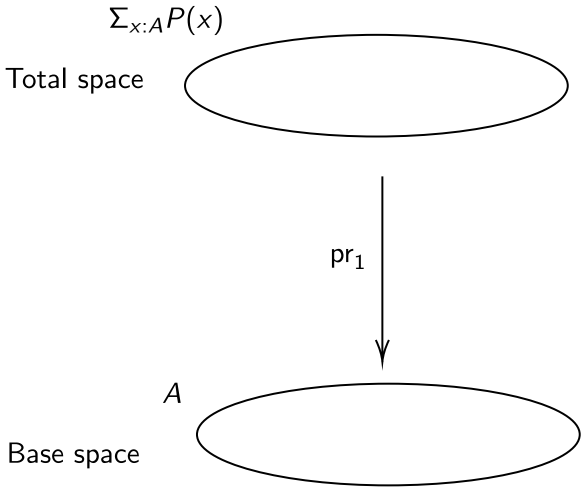
> *We think of a type family $P : A \to \mathcal{U}$ as a fibration with base space A, with $P(x)$ being the fiber over x, and with $\Sigma_{(x:A)}P(x)$ being the total space of the fibration, with first projection $\Sigma_{(x:A)}P(x) \to A$. The defining property of a fibration is that given a path $p : x = y$ in the base space A and a point $u : P(x)$ in the fiber over x, we may lift the path p to a path in the total space starting at u (and this lifting can be done continuously). (§2.3)*

Type Families as Fibrations



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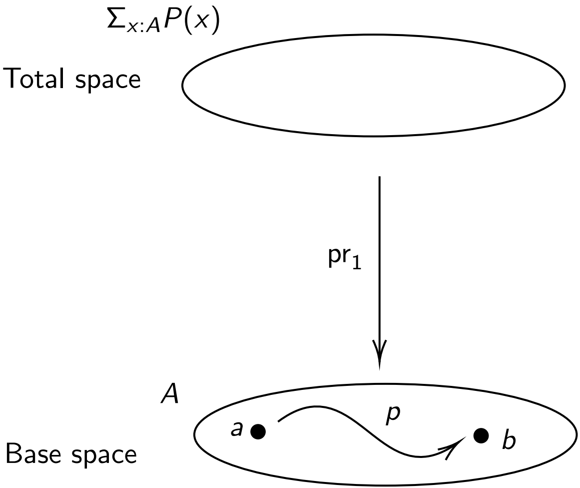

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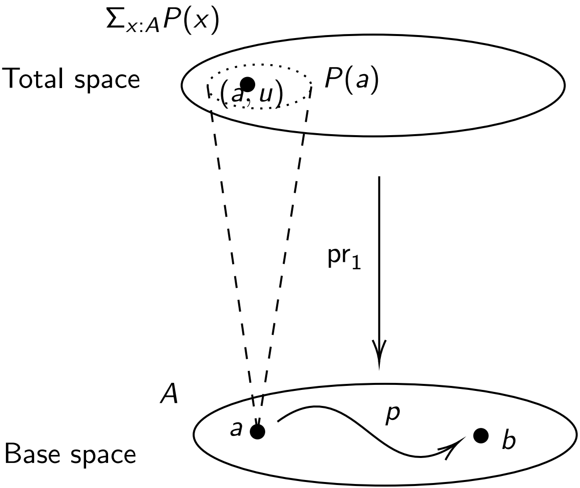

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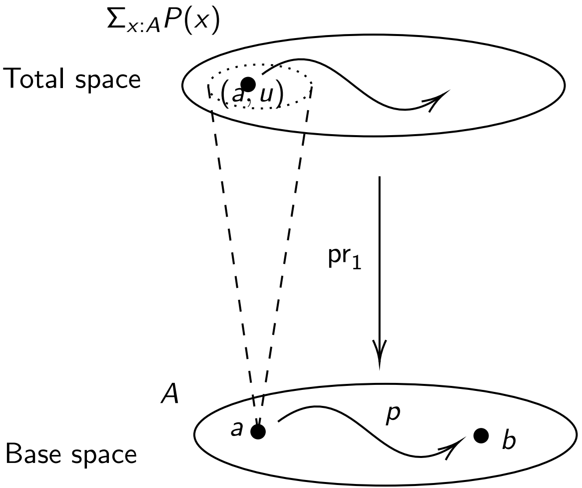

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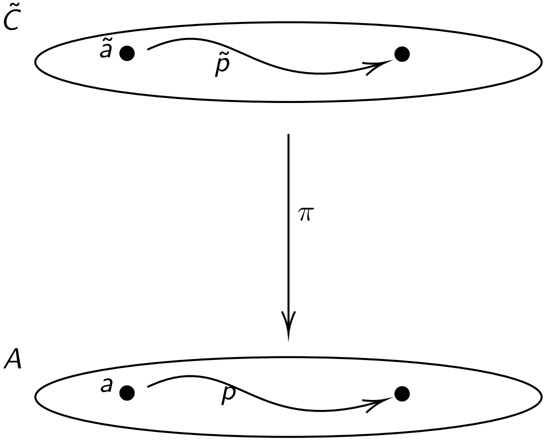
Let A be a topological space with a point a. Let $p : [0,1] \to A$ be a path such that $p(0) = a$. Given a map $\pi : \tilde{C} \to A$ such that there is a point $\tilde{a} \in \tilde{C}$, where $\pi(\tilde{a}) = a$, we say that π has the **path lifting property with respect to** p just in case there is a path $\tilde{p} : [0,1] \to \tilde{C}$ such that $\pi \circ \tilde{p} = p$.

Path Lifting Property



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Path induction follows from path lifting.

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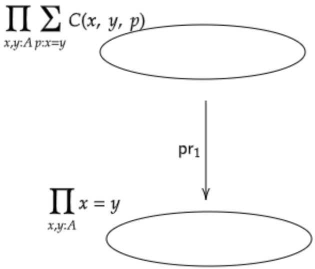
Path Lifting to Path Induction (cont.)



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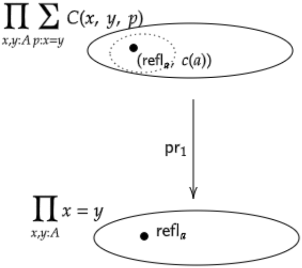

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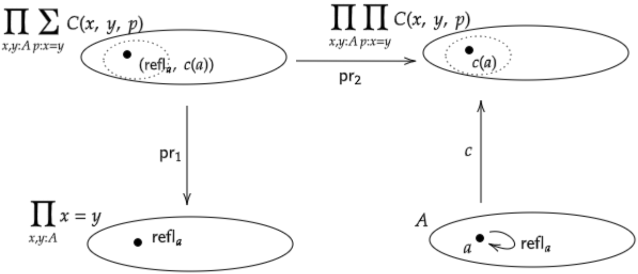

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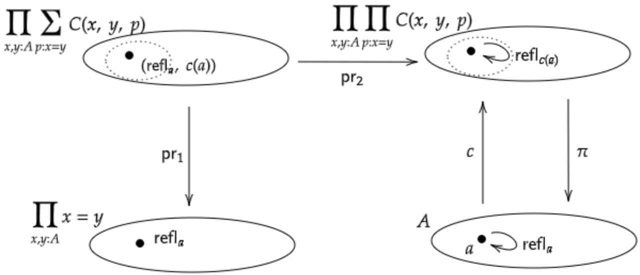

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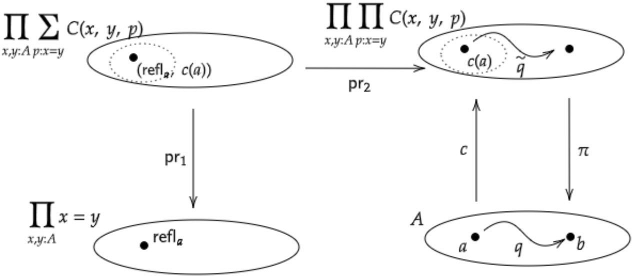

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Thank you.

- Ladyman and Presnell. 2015. "Identity in Homotopy Type Theory, Part I: The Justification of Path Induction." Philosophia Mathematica. Series III 23 (3): 386–406.
- Walsh, Patrick. 2017. "Categorical Harmony and Path Induction." Review of Symbolic Logic 10 (2): 301–21.