Smash Products Are Symmetric Monoidal in HoTT

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Introduction

- The smash product plays a crucial role in homotopy (type) theory
- Key property: it is **symmetric monoidal**
- This 'fact' is useful when doing HoTT too:
  - Brunerie (2016): $\pi_4(S^3) \cong \mathbb{Z}/2\mathbb{Z}$
  - Van Doorn (2018): Cohomological spectral sequences
- Problem: this fact has never been proved in HoTT
- Today: A solution using a 'new' heuristic for reasoning about smash products

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$$\begin{array}{ccc} A \vee B & \longrightarrow & A \times B \\ \downarrow & & \downarrow \\ 1 & \longrightarrow & A \wedge B \end{array}$$

The pentagon

Fact

The smash product is associative. We use $\alpha_{A,B,C} : (A \wedge B) \wedge C \xrightarrow{\sim} A \wedge (B \wedge C)$ to denote the associator.

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- The 'impossible' pentagon axiom for $\wedge$:

$$\begin{array}{ccc}
 & ((A \wedge B) \wedge C) \wedge D & \\
\swarrow^{\alpha_{A,B,C} \wedge 1_D} & & \searrow^{\alpha_{A \wedge B, C, D}} \\
(A \wedge (B \wedge C)) \wedge D & & (A \wedge B) \wedge (C \wedge D) \\
\downarrow {\scriptstyle \alpha_{A, B \wedge C, D}} & & \downarrow {\scriptstyle \alpha_{A,B,C \wedge D}} \\
A \wedge ((B \wedge C) \wedge D) & \xrightarrow{\quad 1_A \wedge \alpha_{B,C,D} \quad} & A \wedge (B \wedge (C \wedge D))
\end{array}$$

The pentagon

- Why is it so hard to verify?
- Proving it amounts to constructing a homotopy

$$(x : ((A \wedge B) \wedge C) \wedge D) \to f\,x = g\,x$$

for the pentagonators f and g.

Induction hell

```
pf : (x : ((A ∧· B) ∧· C) ∧ D)  → f x ≡ g x
pf ⋆ = { }0
pf ⟨ ⋆ , d ⟩ = { }1
pf ⟨ ⟨ ⋆ , c ⟩ , d ⟩ = { }2
pf ⟨ ⟨ ⟨ a , b ⟩ , c ⟩ , d ⟩ = { }3
pf ⟨ ⟨ pushₗ a i , c ⟩ , d ⟩ = { }4
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pf ⟨ ⟨ pushₗᵣ i j , c ⟩ , d ⟩ = { }6
pf ⟨ pushₗ ⋆ i , d ⟩ = { }7
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pf ⟨ pushₗ (pushₗᵣ i j) k , d ⟩ = { }11
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pf (pushₗ (pushₗ ⋆ i₁) i) = { }20
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pf (pushₗ (pushₗ (pushₗᵣ l k) j) i) = { }24
pf (pushₗ (pushᵣ b j) i) = { }25
pf (pushₗ (pushₗᵣ k j) i) = { }26
pf (pushᵣ b i) = { }27
pf (pushₗᵣ i j) = { }28
```

A first step

- **Need:** a better way to deal with equalities of functions $f : \bigwedge_i A_i \to B$

Lemma 2

To check $f = g$ for $f, g : A \wedge B \to C$, the coherence for $\mathsf{push}_{\mathsf{lr}}$ is automatic.

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- Still: 22 (highly non-trivial) cases left...

Interlude: homogeneous types

Definition 3

A pointed type A is homogeneous if for every $a : A$, there is an automorphism $e_a : A \simeq A$ such that $e_a \star_A = a$

- All (pointed) path spaces are homogeneous.

Lemma 4 (Evan's Trick)

Let $f, g : A \to_\star B$ be two pointed functions with B homogeneous. If there is a homotopy $(x : A) \to f\,x = g\,x$, then $f = g$ ***as pointed functions****.*

Interlude: homogeneous types

Lemma 5 (Evans's trick 2.0)

Let $f, g : A \wedge B \to_\star C$ be two pointed functions with C homogeneous. If there is a homotopy

$$((x, y) : A \times B) \to f\langle x, y\rangle = g\langle x, y\rangle$$

then $f = g$ (as pointed functions)

Proof.

Using the adjunction $(A \wedge B \to_\star C) \simeq A \to_\star (B \to_\star C)$. □

- Dream: Apply the trick to pentagon.

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- ~~Dream: Apply the trick to pentagon.~~
- Nightmare: We can't (the codomain is **not** homogeneous).

The heuristic

- Fortunately, there is still hope: loop spaces are homogeneous. Let's 'make them appear' in the proof of the pentagon.

Definition 6

Let $f, g : A \wedge B \to_\star C$. A homotopy $h : ((a, b) : A \times B) \to f\langle a, b\rangle = g\langle a, b\rangle$ induces two functions

- $L_h : A \to \Omega C$
- $R_h : B \to \Omega C$

- For instance, $L_h\, a$ is defined by the composition

$$\star_C \xrightarrow{\star_f^{-1}} f\star_\wedge \xrightarrow{\mathsf{ap}_f(\mathsf{push_l}\ a)^{-1}} f\langle a, \star_B\rangle \xrightarrow{h(a,\star_B)} g\langle a, \star_B\rangle \xrightarrow{\mathsf{ap}_g(\mathsf{push_l}\ a)} g\star_\wedge \xrightarrow{\star_g} \star_C$$

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Lemma 7

If $L_h = \text{const}_{(L_h \star_A)}$ *and* $R_h = \text{const}_{(R_h \star_B)}$, *then* $f = g$

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- $L_h : A \to \Omega C$
- $R_h : B \to \Omega C$

- The point: applying this construction to the pentagonators $f, g : ((A \wedge B) \wedge C) \wedge D \to A \wedge (B \wedge (C \wedge D))$, the functions L_h is of type

$$L_h : (A \wedge B) \wedge C \to \Omega(A \wedge (B \wedge (C \wedge D)))$$

Homogeneous codomain!

The heuristic

- We want to prove that L_h is constant. This is precisely where the explosion of complexity happens in a naive proof...
- ...but thanks to our set up: enough to show that

$$A \times B \times C \xrightarrow{\langle -,-,- \rangle} (A \wedge B) \wedge C \xrightarrow{L_h} \Omega(A \wedge (B \wedge (C \wedge D)))$$

  is constant.
- Amounts to checking the actions of f and g on $\mathsf{push_l}\langle a, b, c \rangle$, but no further coherences!
  - In particular: no nestled $\mathsf{push_l}$ and $\mathsf{push_r}$ constructors.
  - Only ~~13 cases~~ 1 case to check

The heuristic

- By iterating the argument, we may use L_h and R_h to construct equalities $f = g$ for any $f, g : \bigwedge_{i \leq n} A_i \to B$.
- **Heuristic**: We only need to construct a homotopy $h : f\langle x_1, \ldots, x_n\rangle = g\langle x_1, \ldots, x_n\rangle$ and show that it is compatible with ap_f and ap_g on *single* applications of $\mathsf{push_l}$ and $\mathsf{push_r}$.
- Number of cases: ~~$O(2^n)$~~ $O(2n)$

Reaping the fruits

Theorem 8

The smash product satisfies the pentagon identity.

Proof.

After applying of the heuristic, the remaining coherences are easily verified by hand. □

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Proof.

After applying of the heuristic, the remaining coherences are easily verified by hand. □

Theorem 9

The smash product is symmetric monoidal with the booleans as unit.

Thanks

Thanks for listening!

$$f = g$$

$$\downarrow \qquad \searrow$$

$$h : \begin{cases} ((\bar{x}, x_n) : (\bigwedge_{i<n} A_i) \times A_n) \\ \to f\langle \bar{x}, x_n \rangle = g\langle \bar{x}, x_n \rangle \end{cases} \qquad \begin{cases} L_h \langle x_1, \dots\, x_{n-1} \rangle = \text{const} \\ R_h\, x_n = \text{const} \end{cases}$$

$$\downarrow \qquad \searrow$$

$$h_n : \begin{cases} ((\bar{x}, x_{n-1}) : (\bigwedge_{i<n-1} A_i) \times A_{n-1}) \\ \to f\langle \bar{x}, x_{n-1}, x_n \rangle = g\langle \bar{x}, x_{n-1}, x_n \rangle \end{cases} \qquad \begin{cases} L_{h_n} \langle x_1, \dots\, x_{n-2} \rangle = \text{const} \\ R_{h_n}\, x_{n-1} = \text{const} \end{cases}$$

$$\downarrow \qquad \searrow$$

$$\vdots \qquad \qquad \vdots$$

$$\downarrow$$

$$f\langle x_1, \dots, x_n \rangle = g\langle x_1, \dots, x_n \rangle$$