Computing Cohomology Rings in Cubical Agda

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1. Cohomology

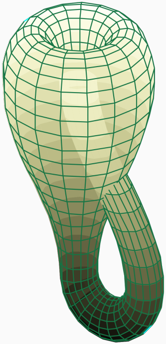
1. Cohomology

1.1 Cohomology Groups

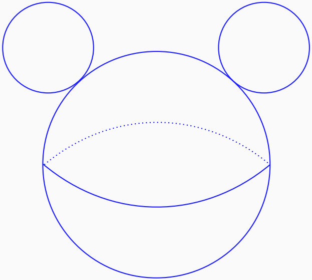
Klein Bottle vs Mickey Mouse

Question :

- How to prove that two topological spaces are not isomorphic ?



Klein Bottle $\mathbb{K}^2$



"Mickey Mouse space"

$\mathbb{S}^2 \bigvee \mathbb{S}^1 \bigvee \mathbb{S}^1$

Question :

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The idea behind cohomology groups

Cohomology Groups

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- To each topological space X, we associate a sequence of abelian groups $(H^i(X))_{i:\mathbb{N}}$, named the cohomology groups, such that:

$$\exists i \in \mathbb{N},\ H^i(X) \ncong H^i(Y) \implies X \ncong Y$$

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- The invariants are supposed to be "easy" to compute, and "nice" groups : $\mathbb{Z},\ \mathbb{Z} \times \mathbb{Z},\ \mathbb{Z}/2\mathbb{Z}$

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- Definitions are synthetic:
  - ▷ No points !
  - ▷ Simple and short computations of many cohomology groups
  - ▷ Functions compute, at least in theory

1. Cohomology

1.2 Cohomology Rings

Cohomology groups are not enough !

Cohomology groups are just invariants

- Some topological spaces are not isomorphic but they have the same cohomology groups

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The Cohomology Ring

The cup product and the comology ring

- There is a graded operation on the groups, the cup product:

$$\smile : \; H^i(X) \longrightarrow H^j(X) \longrightarrow H^{i+j}(X)$$

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What do we need to solve ?

Objective ?

Prove ring isomorphisms of the form :

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- We want to be constructive

2. Building the direct sum and graded rings

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2.1 Adapting the classical direct sum

Adapting the classical direct sum

The classical direct sum

$$\bigoplus_{i:I} G_i := \{(g_i)_{i\in I} \mid \exists \text{ finite } J \subset I, \forall n \notin J, \quad g_n = 0 \in G_n\}$$

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A general definition ?

$$\sum_{f:\prod_{i:I}:G_i} \| \sum_{\substack{J:\mathrm{subset}(I)\\ J \text{ finite}}} \prod_{\substack{i:I\\ i\notin J}} f(i) \equiv 0_i \|$$

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A solution when I is $\mathbb{N}$

$$\bigoplus_{n:\mathbb{N}}^{\mathrm{Fun}} G_n := \sum_{f:\prod_{n:\mathbb{N}}:G_n} \| \sum_{k:\mathbb{N}} \prod_{\substack{k:\mathbb{N} \\ k<i}} f(i) \equiv 0_i \|$$

Building Graded Rings ?

Abelian group structure

Given $f, g : \bigoplus^{\mathrm{Fun}}_{n:\mathbb{N}} G_n$, an abelian group structure can be defined pointwise:

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A product ?

Given $\star : G_i \to G_j \to G_{i+j}$ over $(\mathbb{N}, 0, +)$, we would like to define :

$$(f \times g)(n) = \sum_{i=0}^{n} f(i) \star g(n-i)$$

but it doesn't type check because $G_{i+(n-i)} \neq G_n$ <u>definitionally</u>

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Transports are needed

$$(f \times g)(n) = \sum_{i=0}^{n} \uparrow_i^n (f(i) \star g(n-i))$$

Proving the properties ?

Proving associativity is however complicated, it unfolds to proving :

$$
\begin{aligned}
(f \times (g \times h))(n) &= \sum_{i=0}^{n} \uparrow_i^n (f(i) \star (g \times h)(n-i)) \\
&= \sum_{i=0}^{n} \uparrow_i^n \left(f(i) \star \left(\sum_{j=0}^{n-i} \uparrow_j^{n-i} (\ g(j) \star h(n-i-j)\ ) \right) \right) \\
&\equiv \ldots \\
&= \sum_{i=0}^{n} \uparrow_i^n \left(\left(\sum_{j=0}^{i} \uparrow_j^{i} (\ f(j) \star g(i-j)\ ) \right) \star h(n-i) \right) \\
&= \sum_{i=0}^{n} \uparrow_i^n ((f \times g)(i) \star h(n-i)) \\
&= ((f \times g) \times h)(n)
\end{aligned}
$$

2. Building the direct sum and graded rings

2.2 A quotient inductive type definition

A quotient inductive type definition

```
data ⊕HIT (I : Type) (G : I → AbGroup) : Type where
  -- Point constructors
  0⊕       : ⊕HIT I G
  base     : (n : I) → ⟨ G n ⟩ → ⊕HIT I G
  _+⊕_     : ⊕HIT I G → ⊕HIT I G → ⊕HIT I G
  -- Abelian group laws
  +⊕Assoc : ∀ x y z → x +⊕ (y +⊕ z) ≡ (x +⊕ y) +⊕ z
  +⊕Rid    : ∀ x → x +⊕ 0⊕ ≡ x
  +⊕Comm  : ∀ x y → x +⊕ y ≡ y +⊕ x
  -- Morphism laws
  base0⊕   : ∀ n → base n 0⟨ G n ⟩ ≡ 0⊕
  base+⊕   : ∀ n x y → base n x +⊕ base n y ≡ base n (x +⟨ G n ⟩ y)
  -- Set truncation
  trunc    : isSet (⊕HIT I G)
```

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  $$\text{base}\,(n+(m+k))\ \ (x\star(y\star z)) \equiv \text{base}\,((n+m)+k)\ \ ((x\star y)\star z)$$

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Cohomology Rings

- This enables to define graded ring, and as such cohomology rings.

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- The elements and the product are intuitive and easy to work with
  - ▷ Elements are generated by $0,\ aX^n,\ +$
  - ▷ The product is basically generated by $aX^n \times bX^m = abX^{n+m}$

An application of the SIP

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1. Prove that the raw rings of $\bigoplus^{\text{HIT}}$ and $\bigoplus^{\text{Fun}}$ are equal
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Structural Identity Principle

- By the structural identity principle, it suffices to prove that the raw ring structures are isomorphic as raw rings i.e. as "rings".

3. Proving the isomorphisms ?

A General Method

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The method in practice

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- ψ cancels on $X^2,\ XY,\ 2Y,\ Y^2$ by definition

Using computation to caracterise the cup product

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- And we actually want to compute $\phi_1(1) \smile \phi_1(1)$

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Compute $\alpha \smile \alpha$? Nice try!

- I stopped the computation after 2 minutes and 3gb of ram (note that α is just 8 lines when normalized...)

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