Computing Cohomology Rings in Cubical Agda

Thomas Lamiaux, Axel Ljungström, Anders Mörtberg
HoTT/UF 2023
1. Cohomology
1. Cohomology

1.1 Cohomology Groups
Klein Bottle vs Mickey Mouse

**Question:**

- How to prove that two topological spaces are not isomorphic?

Klein Bottle $K^2$

"Mickey Mouse space" $S^2 \vee S^1 \vee S^1$
Cohomology Groups

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The idea behind cohomology groups
Cohomology Groups

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• To each topological space $X$, we associate a sequence of abelian groups $(H^i(X))_{i : \mathbb{N}}$, named the cohomology groups, such that:

$$\exists i \in \mathbb{N}, \quad H^i(X) \not\cong H^i(Y) \implies X \not\cong Y$$

• The invariants are supposed to be “easy” to compute, and “nice” groups: $\mathbb{Z}$, $\mathbb{Z} \times \mathbb{Z}$, $\mathbb{Z}/2\mathbb{Z}$.
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Working in HoTT

Cohomology groups in HoTT

We are working in synthetic mathematics, in HoTT where cohomology groups have a remarkably short definition:

\[ H^i(X) := \| X \rightarrow \| S^i \| \|_0 \]
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• Definitions are synthetic:
  ▶ No points!
  ▶ Simple and short computations of many cohomology groups
  ▶ Functions compute, at least in theory
1. Cohomology

1.2 Cohomology Rings
Cohomology groups are not enough!

Cohomology groups are just invariants

- Some topological spaces are not isomorphic but they have the same cohomology groups

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The Cohomology Ring

The cup product and the cohomology ring

- There is a graded operation on the groups, the cup product:

  \[ \smile : H^i(X) \to H^j(X) \to H^{i+j}(X) \]

which turns \( H^*(X) := \bigoplus_{i \in \mathbb{N}} H^i(X) \) in a ring named the cohomology ring.
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- This cohomology ring is one more invariant:
  \[ H^*(X) \not\cong H^*(Y) \implies X \not\cong Y \]
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What do we need to solve?

Objective?

Prove ring isomorphisms of the form:

\[ H^*(\mathbb{K}^2) := \bigoplus_{i: \mathbb{N}} H^i(\mathbb{K}^2) \cong \mathbb{Z}[X, Y]/(X^2, XY, 2Y, Y^2) \]
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1. Build an easy to work with \( \bigoplus \) to define graded rings
2. Find a practical notion of multivariate polynomials

Constraints?

- We are working in cubical agda \( \Rightarrow \) no tactics
- We want to be constructive
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2. Building the direct sum and graded rings
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2.1 Adapting the classical direct sum
Adapting the classical direct sum

The classical direct sum

\[ \bigoplus_{i:I} G_i := \{(g_i)_{i \in I} \mid \exists \text{ finite } J \subset I, \forall n \notin J, \ g_n = 0 \in G_n\} \]
Adapting the classical direct sum

The classical direct sum

\[ \bigoplus_{i \in I} G_i := \left\{ (g_i)_{i \in I} \mid \exists \text{ finite } J \subset I, \forall n \notin J, \ g_n = 0 \in G_n \right\} \]

A general definition?

\[ \sum_{f : \prod_{i \in I} G_i} \| \sum_{J : \text{subset}(I)} \prod_{i \in I \setminus J} f(i) \equiv 0_i \| \]

\[ \text{if } (i_j) \equiv 0_i \| \]
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A solution when \( I \) is \( \mathbb{N} \)

\[ \bigoplus_{n: \mathbb{N}} G_n := \sum_{f: \prod_{n: \mathbb{N}} G_n} \parallel \sum_{k: \mathbb{N}} \prod_{k: \mathbb{N}, k < i} f(i) \equiv 0_i \parallel \]
### Abelian group structure

Given $f, g : \bigoplus_{n : \mathbb{N}} G_n$, an abelian group structure can be defined pointwise:

$$(f + g)(n) = f(n) +_n g(n)$$
Abelian group structure

Given $f, g : \bigoplus_{n : \mathbb{N}}^\text{Fun} G_n$, an abelian group structure can be defined pointwise:

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A product ?

Given $\star : G_i \to G_j \to G_{i+j}$ over $(\mathbb{N}, 0, +)$, we would like to define:

$$(f \times g)(n) = \sum_{i=0}^{n} f(i) \star g(n - i)$$

but it doesn’t type check because $G_{i+(n-i)} \neq G_n$ definitionally
Building Graded Rings

**Abelian group structure**

Given \( f, g : \bigoplus_{n: \mathbb{N}}^\text{Fun} G_n \), an abelian group structure can be defined pointwise:

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**Transports are needed**

\[
(f \times g)(n) = \sum_{i=0}^{n} \uparrow_i^n (f(i) \star g(n - i))
\]
Proving the properties?

Proving associativity is however complicated, it unfolds to proving:

\[(f \times (g \times h))(n) = \sum_{i=0}^{n} \uparrow_i^n (f(i) \star (g \times h)(n - i))\]

\[= \sum_{i=0}^{n} \uparrow_i^n \left( f(i) \star \left( \sum_{j=0}^{n-i} \uparrow_{j}^{n-i} (g(j) \star h(n - i - j)) \right) \right)\]

\[\equiv ...\]

\[= \sum_{i=0}^{n} \uparrow_i^n \left( \left( \sum_{j=0}^{i} \uparrow_{j}^{i} (f(j) \star g(i - j)) \right) \star h(n - i) \right)\]

\[= \sum_{i=0}^{n} \uparrow_i^n (\left( f \times g \right)(i) \star h(n - i))\]

\[= \left( (f \times g) \times h \right)(n)\]
2. Building the direct sum and graded rings

2.2 A quotient inductive type definition
A quotient inductive type definition

data ⊕HIT (I : Type) (G : I → AbGroup) : Type where
  -- Point constructors
  0⊕ : ⊕HIT I G
  base : (n : I) → ⟨ G n ⟩ → ⊕HIT I G
  _++_ : ⊕HIT I G → ⊕HIT I G → ⊕HIT I G
  -- Abelian group laws
  ⊕Assoc : ∀ x y z → x ++ (y ++ z) ≡ (x ++ y) ++ z
  ⊕Rid : ∀ x → x ++ 0⊕ ≡ x
  ⊕Comm : ∀ x y → x ++ y ≡ y ++ x
  -- Morphism laws
  base0⊕ : ∀ n → base n 0⟨ G n ⟩ ≡ 0⊕
  base+_⊕ : ∀ n x y → base n x ++ base n y ≡ base n (x ++ ⟨ G n ⟩ y)
  -- Set truncation
  trunc : isSet (⊕HIT I G)
Defining a graded ring

Defining the product

Given a monoid \((I, e, +)\) and \(\star : G_i \rightarrow G_j \rightarrow G_{i+j}\), we can define a product \(_\times _\) by double recursion:

• To base \(n \times m\), we associate base \((n + m) \times (x \star y)\).
• The other cases are trivial.

Proving associativity

We can again reason by triple induction:

• The base case unfolds to proving:
  \[\text{base}(n + (m + k) \times (x \star (y \star z))) \equiv \text{base}((n + m) + k \times (x \star y) \star z)\]
• The other cases are trivial.

Cohomology Rings

This enables to define graded rings, and as such cohomology rings.
Defining the product

Given a monoid \((I, e, +)\) and \(\star : G_i \to G_j \to G_{i+j}\), we can define a product \(\_ \times \_\) by double recursion:

- To base \(n \times\), base \(m \times\), we associate base \((n + m) \times (x \star y)\)

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  - The other cases are trivial

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This enables to define graded ring, and as such cohomology rings.
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The HIT polynomials

Multivariate polynomials

- We can define $R[X] := \bigoplus_{i \in \mathbb{N}}^{\text{HIT}} R$

Pros for our purpose

- This is a direct definition of multivariate polynomials
- The elements and the product are intuitive and easy to work with
- Elements are generated by $0$, $aX^n$, $+$
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**Transporting the properties**

1. Prove that the raw rings of \(\bigoplus^\text{HIT}\) and \(\bigoplus^\text{Fun}\) are equal
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Structural Identity Principle

- By the structural identity principle, it suffices to prove that the raw ring structures are isomorphic as raw rings i.e. as "rings".
3. Proving the isomorphisms?
A General Method

Objective?

Prove ring isomorphisms of the form:

\[ H^*(\mathbb{K}^2) := \bigoplus_{i \in \mathbb{N}} H^i(\mathbb{K}^2) \cong \mathbb{Z}[X, Y]/\langle X^2, XY, 2Y, Y^2 \rangle \]
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The benefit of the data structure

The method in practice

Thanks to the data structure:

- Building the functions is very direct by induction
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$$\psi(X^mY^n \times X^kY^l) \equiv \psi(X^mY^n) \cup \psi(X^kY^l)$$

i.e. studying the cup product
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i.e. studying the cup product
- $\psi$ cancels on $X^2, XY, 2Y, Y^2$ by definition
Using computation to characterise the cup product

We need to prove that:

\[ \psi(\times) \equiv \psi(\odot) \]

For the Klein bottle

Writing \( \phi_1: \mathbb{Z} \sim H_1(K^2) \) it unfolds to proving:

\[ \phi_1(1) \odot \phi_1(1) = 0 \]

Computing the result?

I stopped the computation of \( \phi_1(1) \) after 10 minutes of computation and 3gb of ram!
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- And we actually want to compute \( \phi_1(1) \cup \phi_1(1) \)
Using computation to characterise the cup product

Work with an alternative generator $\alpha$

1. Define a nicer generator $\alpha : H^1(K^2)$

Prove that $\varphi - 1(\alpha) = 1$ by computation

Conclude that $\alpha = \varphi 1(1)$ for free

Prove that $\alpha \cdot \alpha = 0$ (10 lines of Agda)

Compute $\alpha \cdot \alpha$? Nice try!

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19/20
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More in the CPP's paper

- Computation of the $\mathbb{Z}$ cohomology ring of: $S^n$, $CP^2$, $S^4$, $W_S^2$, $K_2$, $RP^2$, $W_S^1$
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