Higher coherence equations of semi-simplicial types as *n*-cubes of proofs

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Straightforward to do in type theory by restricting ourselves to **h-Set**s

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The semi-simplicial identity $d_i d_j = d_{j-1} d_i$ induces "higher proof terms" that interfere with each other and need to be identified

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- This is given by

$$\pi := \alpha_{j,k} \cdot \alpha_{i,k-1} \cdot \alpha_{i,j} \quad \text{and} \quad \pi' := \alpha_{i,j} \cdot \alpha_{i,k} \cdot \alpha_{j-1,k-1}$$

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- We now need the data of a term $\beta_{i,j,k} : \pi = \pi'$, and so on...

Possible approaches

- *n*-Truncate (e.g. **h-Set**s, **h-Grp**s, …)

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A general solution is believed to be impossible in "plain" HoTT

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2023 *Indexed* + **h-Set**s + *Parametricity* (Herbelin, Ramachandra)

TLCA 2015 — Warsaw, Poland

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Strict composition

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The explicit form taken by the higher-order coherence equations is well described in the case of (strict) ω-categories, i.e. when associativity and the exchange law hold on the nose for path composition

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- Ian R. Aitchison. The geometry of oriented cubes. Macquarie University Research Report No: 86–0082, 1986.

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- Composition of *n* face maps corresponds to the "spine" of an *n*-cube
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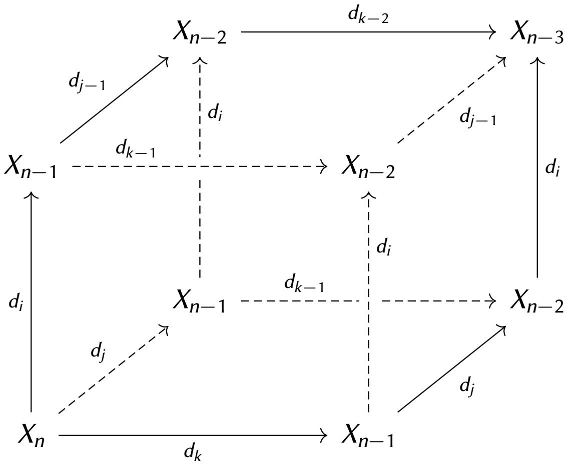
How to give a recursive formulation of all the compositions involved?

3-Cube of proofs



$$\beta_{i,j,k} : \overbrace{\alpha_{j,k} \cdot \alpha_{i,k-1} \cdot \alpha_{i,j}}^{\text{Backmost}} =_{\left(d_i d_j d_k =_{(X_n \to X_{n-3})} d_{k-2} d_{j-1} d_i\right)} \underbrace{\alpha_{i,j} \cdot \alpha_{i,k} \cdot \alpha_{j-1,k-1}}_{\text{Frontmost}}$$

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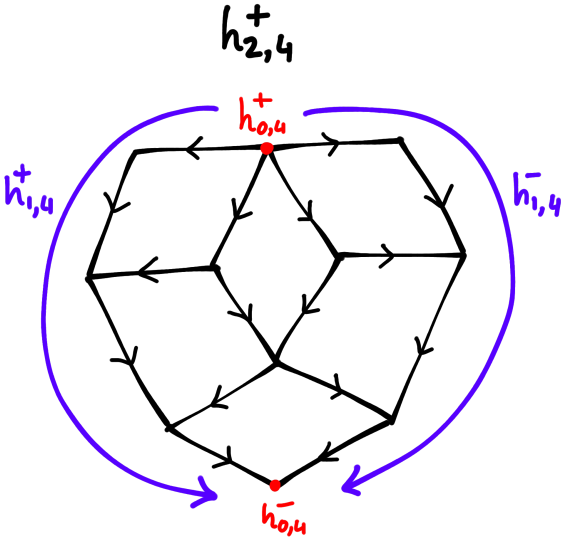
- We write $h^{\pm}_{k,n}$ for the k-hemispheres of the n-cube

Illustration



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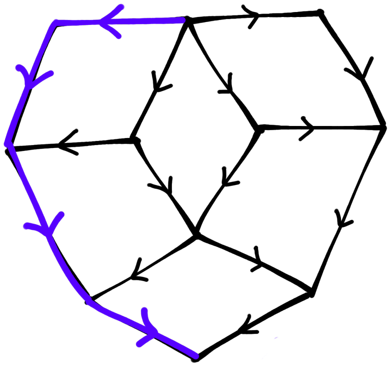

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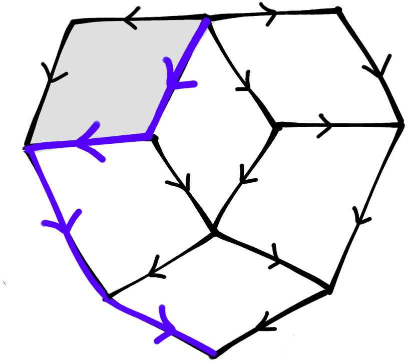

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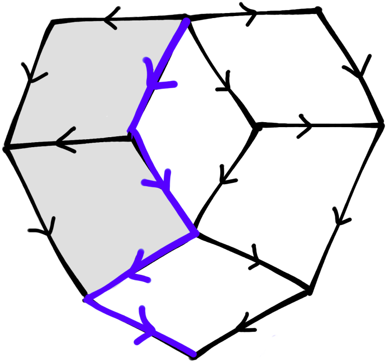

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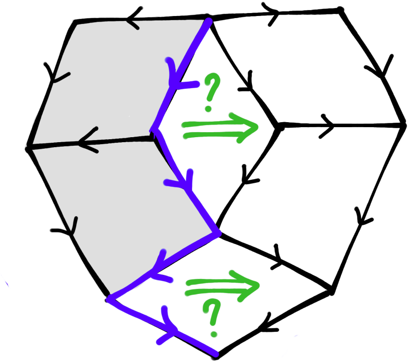

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D_k^n is the poset of *increasing* sequences $x_1 \dots x_n$ of length n with values $0 \leq x_i \leq k$ equipped with the "pointwise" order

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Theorem

The k-hemispheres of the n-cube are described by D_{n-k}^k in the sense that there is a bijection sending any $h_{k,n}^{\pm}$ onto a linear extension of D_{n-k}^k

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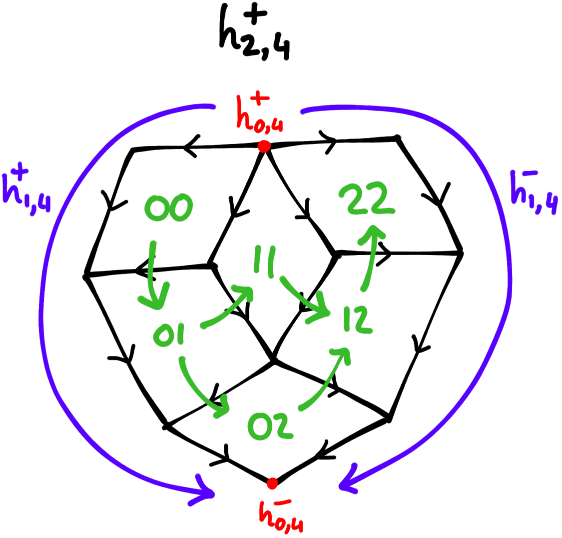

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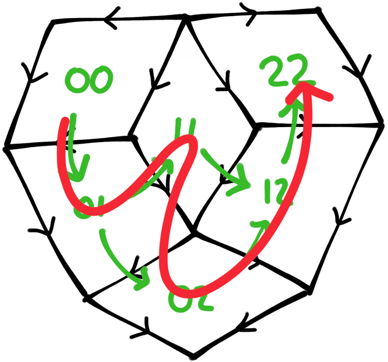

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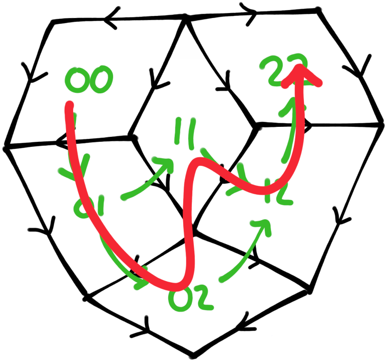

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D_k^n has a rich structure—so does $h_{k,n}^{\pm}$ as a result

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- There is (essentially) one canonical choice of linear extension of D_k^n given by

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- D_k^n can be recursively constructed with maps
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- This construction preserves $\preceq$ defined above!

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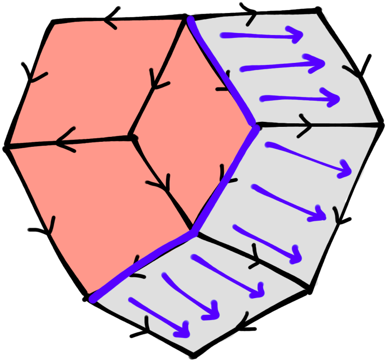
Observation

$h^{\pm}_{k,n}$ has a similar recursive formula

Extrusion of the 3-cube

$$h^{+}_{2,4} = \text{``}h^{+}_{2,3} + h^{-}_{1,3}\text{''}$$



Aitchison's report "only" contains formulae for the k-hemispheres

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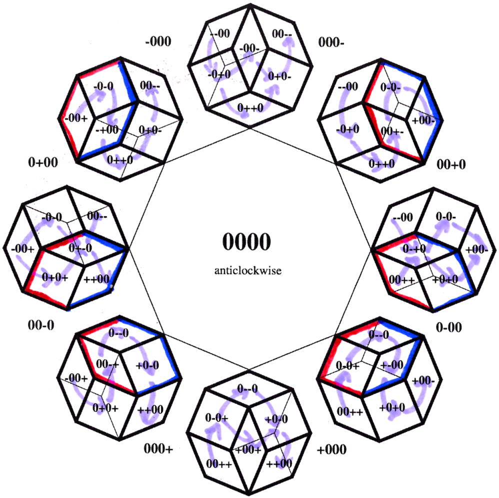
- Give an explicit description of the "internal" compositions, which have a structure different from *k*-hemispheres (see Appendix)
- Make explicit the role of parenthesizing (associativity, exchange law, identity, ...)
- Combine these results to attempt a definition of semi-simplicial types with the coherence equations made explicit at every level

Questions?

Appendix



“Octagon of octagons” — Extracted from Aitchison’s report

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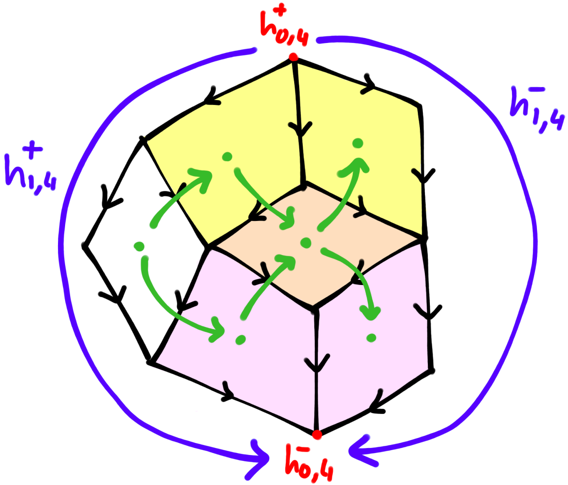

Appendix