A Foundation for Synthetic Algebraic Geometry

Felix Cherubini, Thierry Coquand, Matthias Hutzler

Acknowledgements and Subprojects

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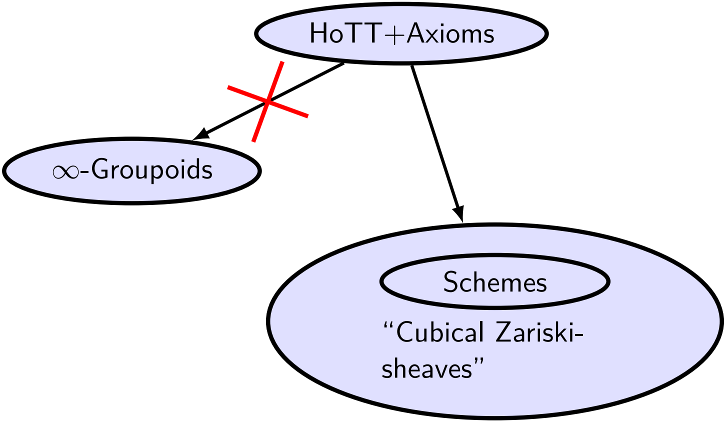
Present work is joint with Ingo Blechschmidt, Hugo Moeneclaey, Josselin Poiret, and David Wärn.

Foundations	(Felix, Matthias, Thierry)
Proper Schemes	(David, Felix, Matthias, Thierry)
Differential Geometry	(David, Felix, Hugo, Matthias)
Čech Cohomology	(David, Felix, Ingo)
Formalization	(Felix, Josselin, Matthias)

`https://github.com/felixwellen/synthetic-zariski/`



* Schemes = quasi-compact, quasi-separated schemes of finite type

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Example: $\mathrm{Spec}(R[X]) = R$. Thus:

$$R[X] \xrightarrow{\sim} R^R$$

polynomials = functions !!

Towards schemes

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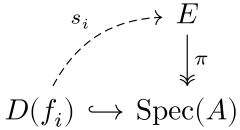
But is it an equivalence??
Yes, using *Zariski-local choice*!

Zariski-local choice

Axiom (Zariski-local choice):
For every surjective π, there merely exist local sections s_i



with $f_1, \dots, f_n : A$ coprime.

Some more results

- $\mathrm{OpenProp}^{\mathrm{Spec}(A)} \cong \{$f.g. radical ideals of $A\}$
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- All functions $\mathrm{Spec}\, A \to \mathbb{N}$ are bounded.
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For $A : X \to \mathrm{Ab}$, define *cohomology* as:

$$H^n(X, A) :\equiv \left\| \prod_{x:X} K(A_x, n) \right\|_{\mathrm{set}}$$

- H^n coincides with Čech-Cohomology (for *separated* schemes).
- A scheme X is affine if and only if

$$H^n(X, M) = 0$$

for all $M : X \to R\text{-}\mathrm{Mod}_{\mathrm{wqc}}$ and $n > 0$.

Thank you!

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We have a Mayer-Vietoris-Lemma and more generally correspondence with Čech-Cohomology, for nice enough spaces.

Zariski-Choice and Cohomology

Let $X = \mathrm{Spec}(A)$ and $M : X \to R$-Mod such that $((x : D(f)) \to M_x) = ((x : X) \to M)_f$, then

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So for $t_{ij}(x) :\equiv s_j(x)^{-1} \cdot s_i(x)$ we have $t_{ij} + t_{jk} = t_{ik}$. By algebra, we get $u_i : (x : D(f_i)) \to M_x$ with $t_{ij} = u_i - u_j$. Then the $\tilde{s}_i :\equiv s_i - u_i$ glues to a global trivialization.