Colimits in the category of pointed types

Perry Hart and Favonia

University of Minnesota

April 23, 2023

Overview

- The ∞-category $\mathcal{U}^*$ of pointed types and *pointed* functions $A \to_* B :\equiv \sum_{f:A\to B} f(a_0) = b_0$ as a useful setting for synthetic homotopy theory
- In particular, a type-theoretic proof of the Brown representability theorem
- In this talk, the construction of all **(homotopy) colimits** in $\mathcal{U}^*$ of diagrams over *strong trees*.

  This construction takes place in Univ + Pushouts.

Graphs

Consider a graph $\Gamma := (\Gamma_0, \Gamma_1)$ with basepoint $j_0 : \Gamma_0$.

Γ_0 the type of vertices, $\Gamma_1(x, y)$ the type of edges from x to y.

Towards strong trees

We have a type of **zigzags of chains of adjacent edges** in Γ.

$$
\begin{aligned}
\mathcal{Z}_\Gamma^{j_0} &: \Gamma_0 \to \mathcal{U} \\
\mathsf{nil}^{j_0} &: \mathcal{Z}_\Gamma^{j_0}(j_0) \\
\mathsf{right}^{j_0} &: \prod_{i,j:\Gamma_0} \mathcal{Z}_\Gamma^{j_0}(i) \to \Gamma_1(i,j) \to \mathcal{Z}_\Gamma^{j_0}(j) \\
\mathsf{left}^{j_0} &: \prod_{i,j:\Gamma_0} \mathcal{Z}_\Gamma^{j_0}(i) \to \Gamma_1(j,i) \to \mathcal{Z}_\Gamma^{j_0}(j)
\end{aligned}
$$

Assume that the graph Γ is connected in the sense that it has a zigzag

$$\nu_{j_0,i} \ : \ \mathcal{Z}_{\Gamma}^{j_0}(i)$$

from j_0 to i for each vertex $i : \Gamma_0$.

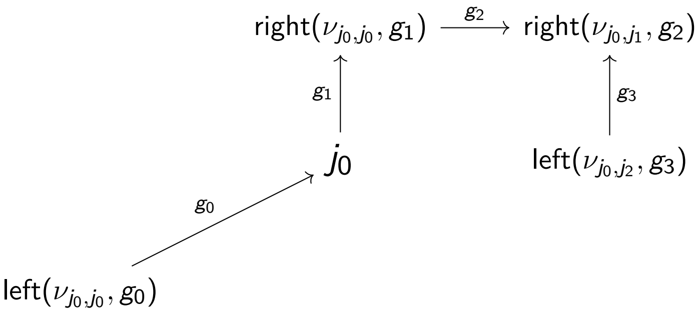
Further, suppose that for all $i, j : \Gamma_0$ and $g : \Gamma_1(i, j)$, the coproduct

$$\left(\nu_{j_0,i} = \mathrm{left}^{j_0}_{j,i}(\nu_{j_0,j}, g)\right) + \left(\nu_{j_0,j} = \mathrm{right}^{j_0}_{i,j}(\nu_{j_0,i}, g)\right)$$

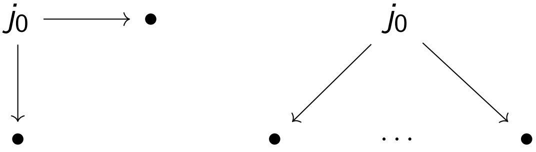
is inhabited.

Intuitively, this means that Γ has no cycles.

In this case, we say that Γ is a **strong tree (at j_0)**, denoted by $\mathrm{StrongTree}(\Gamma)$.


right(ν_{j_0,j_0}, g_1)
g_2
right(ν_{j_0,j_1}, g_2)
g_1
g_3
j_0
left(ν_{j_0,j_2}, g_3)
g_0
left(ν_{j_0,j_0}, g_0)

Examples of strong trees:



$$\mathbf{0} \longrightarrow 1 \longrightarrow 2 \longrightarrow 3 \longrightarrow \cdots$$

$$\cdots \longrightarrow -2 \longrightarrow -1 \longrightarrow \mathbf{0} \longrightarrow 1 \longrightarrow 2 \longrightarrow \cdots$$

A couple of facts:

If Γ_0 is a set and $\Gamma_1(i,j)$ is a set for all $i,j : \Gamma_0$, then $\mathsf{StrongTree}(\Gamma)$ is a mere proposition.

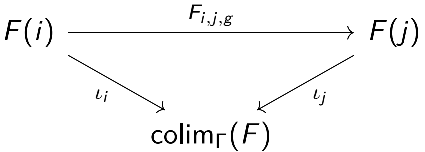
Every strong tree Γ is a tree, i.e., the quotient ${}^{\Gamma_0}\!/\!{}_{\Gamma_1}$ is contractible.

Colimits

Consider a diagram F over Γ together with

- a basepoint b_i of $F(i)$ for each $i : \Gamma_0$
- an identity $p_{i,j,g} : F_{i,j,g}(b_i) = b_j$ for all $i, j : \Gamma_0$ and $g : \Gamma_1(i, j)$.

We can form the colimiting cocone



under F in the unpointed category $\mathcal{U}$.

Suppose that Γ is a strong tree.

By induction on $\mathcal{Z}_\Gamma^{j_0}$, we can find a term

$$\mathrm{ptd} \ : \ \prod_{i:\Gamma_0} \mathcal{Z}_\Gamma^{j_0}(i) \to (\iota_i(b_i) = \iota_{j_0}(b_{j_0}))\,.$$

Thanks to the coherence condition on ν_{j_0}, we have an identity

$$\begin{array}{ccc} (F(i), b_i) & \xrightarrow{\quad (F_{i,j,g}, p_{i,j,g}) \quad} & (F(j), b_j) \\ & {\scriptstyle (\iota_i, \mathrm{ptd}(\nu_{j_0,i}))} \searrow \qquad \swarrow {\scriptstyle (\iota_j, \mathrm{ptd}(\nu_{j_0,j}))} & \\ & (\mathrm{colim}_\Gamma(F), \iota_{j_0}(b_{j_0})) & \end{array}$$

of pointed maps.

Thus, the pointed type $(\mathrm{colim}_\Gamma(F), \iota_{j_0}(b_{j_0}))$ has the structure of a cocone under F in the *pointed* category $\mathcal{U}^*$.

Theorem

If Γ is a strong tree, then the pointed type $\mathrm{colim}_\Gamma(F)$ is the colimit of F in $\mathcal{U}^*$, in the sense that the function

$$(\mathrm{colim}_\Gamma(F) \to_* T) \to \lim_{i:\Gamma^{\mathrm{op}}}(F(i) \to_* T)$$

is an equivalence for every pointed type T.

Corollary

The forgetful functor $\mathcal{U}^* \xrightarrow{\mathtt{pr}_1} \mathcal{U}$ preserves colimits of diagrams over strong trees.

This statement is false when "over strong trees" is removed.

Extending theorem to trees (in progress)

Update $\mathcal{Z}_\Gamma^{j_0} : \Gamma_0 \to \mathcal{U}$ to the indexed HIT generated by

$$\begin{aligned} \mathsf{nil}^{j_0} &: \mathcal{Z}_\Gamma^{j_0}(j_0) \\ \mathsf{cons}^{j_0} &: \prod_{i,j:\Gamma_0} \prod_{g:\Gamma_1(i,j)} \mathcal{Z}_\Gamma^{j_0}(i) \simeq \mathcal{Z}_\Gamma^{j_0}(j). \end{aligned}$$

By Kraus and von Raumer (2019), we have a commuting square

$$\begin{array}{ccc} [j_0] =_{\Gamma_0/\Gamma_1} [i] & \xrightarrow{\simeq} & \mathcal{Z}_\Gamma^{j_0}(i) \\ \downarrow & & \downarrow \mathsf{cons}^{j_0}_{i,j,g} \\ [j_0] =_{\Gamma_0/\Gamma_1} [j] & \xrightarrow[\simeq]{} & \mathcal{Z}_\Gamma^{j_0}(j) \end{array} .$$

It follows that

$$\mathsf{Tree}(\Gamma) \longleftrightarrow \mathsf{StrongTree}(\Gamma)$$

for every graph Γ.

To do:

Check whether the original proof of the theorem respects the new coherence conditions.

Future work

- Agda formalization of main results
- Use theorem to prove Brown representability in type theory.

  Moderately nice functors $(\mathcal{U}^*)^{\mathsf{op}} \to \mathbf{Set}$ *are representable on subuniverses consisting of iterated colimits over strong trees.*