Specifying QIITs using Containers

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Homotopy Type Theory/Univalent Foundations

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Example

```
data Con : Set
data Ty : Con → Set

data Con where
   ◇ : Con
   _,_ : (Γ : Con) (A : Ty Γ) → Con
   eq : (Γ : Con) (A : Ty Γ) (B : Ty (Γ , A)) →
        ((Γ , A) , B) ≡ (Γ , Σ Γ A B)

data Ty where
   ι : (Γ : Con) → Ty Γ
   Σ : (Γ : Con) (A : Ty Γ) → Ty (Γ , A) → Ty Γ
```

Specifications of inductive types

Class of types	**Functor type**	**Category theory semantics**	**Type theoretic normal form**	**Universal type**
ordinary inductive types e.g. $\mathbb{N}$: Set	$\mathbf{Set} \to \mathbf{Set}$	initial algebras of endofunctors on **Set**	containers	W-type
inductive families e.g. $\texttt{Fin} : \mathbb{N} \to \mathrm{Set}$	$(\mathbf{I} \to \mathbf{Set}) \to (\mathbf{I} \to \mathbf{Set})$	initial algebras of endofuntors on $\mathbf{Set}^I$	indexed containers	WI-type

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3. Category of algebras. $\mathbf{A_{n+1}}$ is the category having
   - objects of type $\sum(A : |\mathbf{A_n}|)(c \colon (x : L_n\, A) \to R_n(A, x))$
   - morphisms $(A, c) \to (A', c')$ are morphisms $f \colon A \to A'$ in $\mathbf{A_n}$ such that

$$\begin{array}{ccc} (x : L_n\, A) & \xrightarrow{L_n\, f} & ((L_n\, f)\, x : L_n\, A') \\ {\scriptstyle c}\downarrow & & \downarrow{\scriptstyle c'} \\ R_n(A, x) & \xrightarrow[R_n\, \bar{f}]{} & R_n(A', (L_n\, f)\, x) \end{array}$$

where $\bar{f}$ is the morphism in $\int L_n$ determined by f.

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QIITs e.g. $\mathtt{Con} : \mathrm{Set}$, $\mathtt{Ty} : \mathtt{Con} \to \mathrm{Set}$	sequence of functors L_n and R_n and sequence of categories of dialgebras	initial object in last constructed category of dialgebras $\mathbf{A_n}$	?	?

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Generalised containers

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Definition

A *generalised container* $S \triangleleft P$ over a category $\mathbf{C}$ is a pair $S : \mathrm{Set}$ and $P \colon S \to |\mathbf{C}|$.

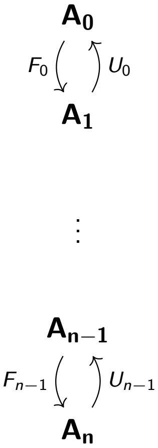
Definition

The *generalised container extension functor* associated to $S \triangleleft P$ and having type $\mathbf{C} \to \mathbf{Set}$, is defined by

$$[\![S \triangleleft P]\!]\, X := \sum (s : S)(\mathbf{C}(P\, s, X))$$

on objects $X : |\mathbf{C}|$.

Illustration



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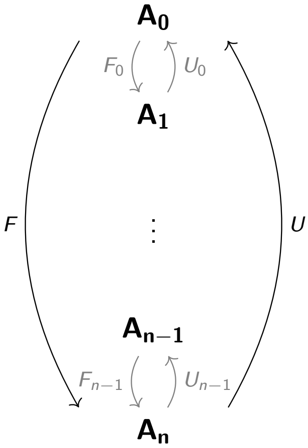

Conclusion

- QIITs combine set-truncated equalities with induction-induction.
- We can represent QIITs semantically as initial dialgebras.
- **Conjecture**: The categories of dialgebras having an initial object are those whose constructors are restricted to **generalised container functors**.
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- **Question**: What does a **universal QW-type** look like?

Thank you!

References

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