UNIVERSITÀ DEGLI STUDI DI MILANO

A 2-dimensional analysis of comprehension

HoTT/UF workshop

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April 22, 2023

Overview

1. Comprehension as an adjoint
2. Comprehension structures
3. Substitution
4. Looking for dimension 2 in dependent types
5. The comprehension biequivalence
6. An application to simplicial sets

Logic and adjunctions

In [Lawvere, 1969] and [Lawvere, 1970] many logical concepts are shown to be part of an adjoint pair.

$$
\begin{aligned}
\text{terminal} &\dashv \top \\
\bot &\dashv \text{terminal} \\
\text{diagonal} &\dashv \wedge \\
\vee &\dashv \text{diagonal} \\
- \wedge A &\dashv A \Rightarrow - \\
\exists &\dashv \text{weakening} \\
\text{weakening} &\dashv \forall
\end{aligned}
$$

Comprehension is an adjoint as well. How?

Logic and adjunctions: how-to

Let $P\colon \mathcal{B}^{\mathrm{op}} \to \mathbf{InfSL}$ an *elementary existential doctrine*, *i.e.*

- $\mathcal{B}$ a category with finite products,
- P a product-preserving functor,

where intuitively $\mathcal{B}$ is the category of contexts and substitutions and on a given Γ, $P(\Gamma)$ is the inf-semilattice of predicates on Γ, such that

(elem) for all Γ there exists $\delta_\Gamma \in P(\Gamma \times \Gamma)$ s.t. for all Θ

$$\text{æ}_{\Theta,\Gamma}\colon P(\Theta \times \Gamma) \longrightarrow P(\Theta \times (\Gamma \times \Gamma))$$
$$A \mapsto P_{\langle \mathrm{pr}_1, \mathrm{pr}_2 \rangle}(A) \wedge P_{\langle \mathrm{pr}_2, \mathrm{pr}_3 \rangle}(\delta_\Gamma)$$

is *left adjoint* to $P_{\langle \mathrm{pr}_1, \mathrm{pr}_2, \mathrm{pr}_2 \rangle}$, and

(ex) for all $\sigma : \Theta$, the reindexing P_σ has a *left adjoint* $\exists_\sigma$

+ naturality + coherence.

First-order logic

Example (Tarski-Lindenbaum doctrine)

Let $\mathcal{T}$ be a first-order theory in a language $\mathcal{L}$ with variables V. Consider $\mathbf{ctx}$ of variables $x = (x_1, \ldots, x_n)$ and substitutions $[t_1/y_1, \ldots, t_m/y_m] = [t/y] \colon x \to y$ and the functor $LT_{\mathcal{T}} \colon \mathbf{ctx}^{\mathrm{op}} \to \mathbf{InfSL}$

$$LT_{\mathcal{T}} \colon x \mapsto \{\text{wff formulae with free (at most) } x\}/_{\dashv\vdash_{\mathcal{T}}}$$

$$A \longmapsto P_{\langle \mathrm{pr}_1, \mathrm{pr}_2 \rangle}(A) \wedge P_{\langle \mathrm{pr}_2, \mathrm{pr}_3 \rangle}(\delta)$$

$$P(y,x) \underset{P_{\langle \mathrm{pr}_1, \mathrm{pr}_2, \mathrm{pr}_2 \rangle}}{\overset{æ_{y,x}}{\underset{\leftarrow}{\overset{\rightarrow}{\bot}}}} P(y,x,x)$$

$$y, x \vdash A(y,x) \rightsquigarrow y, x, x' \vdash A(y,x) \wedge \delta(x, x')$$

and

$y, x, x'; A(y,x) \wedge \delta(x,x') \vdash B(y,x,x')$ iff
$y, x; A(y,x) \vdash B(y,x,x)$

First-order logic

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$$LT_{\mathcal{T}} \colon x \mapsto \{\text{wff formulae with free (at most) } x\}/_{\dashv\vdash_{\mathcal{T}}}$$

$$A \longmapsto \exists y.A$$

$$P(y,x) \underset{P_{\mathrm{pr}_2}}{\overset{\exists_{\mathrm{pr}_2}}{\rightleftarrows}} P(x) \qquad (\exists_{\mathrm{pr}_2} \dashv P_{\mathrm{pr}_2})$$

$y, x \vdash A(y,x) \rightsquigarrow x \vdash \exists y.A(y,x)$

and

$x; \exists y.A(y,x) \vdash B(x)$ iff

$y, x; A(y,x) \vdash B(x)$

The comprehension adjunction

Let $P\colon \mathcal{B}^{\mathrm{op}} \to \mathbf{InfSL}$ an elementary existential doctrine. Then one can define

$$\mathcal{B}_{/\Gamma} \to P(\Gamma)\colon \quad \Theta \xrightarrow{\sigma} \Gamma \ \mapsto\ \exists_\sigma(1_\Theta)\,.$$

Example (Subsets)

Consider the eed $Sub\colon \mathbf{Set}^{\mathrm{op}} \to \mathbf{InfSL}$, $A \mapsto 2^A$.

$$\mathbf{Set}_{/A} \to 2^A\colon \quad B \xrightarrow{f} A \ \mapsto\ \exists_f(1_B) = \bar{f}$$

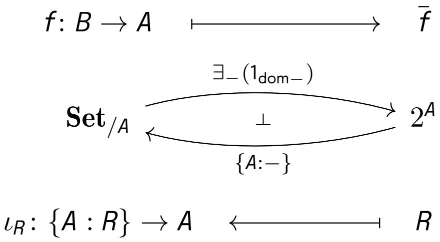
where $\bar{f}(a) = 1$ iff $a \in \mathrm{Im}(f)$

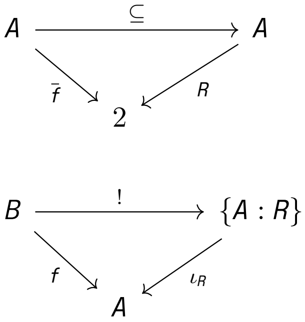
Definition (Comprehension schema)

An eed satisfies the *comprehension schema* if for all Γ the functor above has a right adjoint $\{\Gamma : -\}$ which is natural in Γ.

Proposition

The subset doctrine satisfies the comprehension schema.



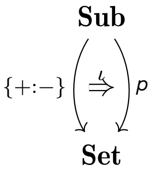


*we abuse the notation $\{A : -\}$ a bit

If computing $\bar{f}$ produces $\bar{f}(a) = 1$ iff $a \in \mathrm{Im}(f)$, then $\{A : R\} = \{a \in A \mid R(a) = 1\}$.

Comprehension structures

Instead of:
$Sub\colon \mathbf{Set}^{\mathrm{op}} \to \mathbf{InfSL}$ and $\{\Gamma : -\}\colon P(\Gamma) \to \mathcal{B}_{/\Gamma}$ natural in Γ,
consider
$p\colon \mathbf{Sub} \to \mathbf{Set}$[1] and $\{+ : -\}, \iota$.



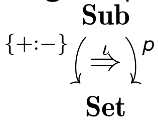
Definition ([Melliès and Rolland, 2020])

A *comprehension structure* on a functor $p\colon \mathcal{E} \to \mathcal{B}$ is a pair $\{+ : -\}, \iota$ with $\{+ : -\}\colon \mathcal{E} \to \mathcal{B}$ a functor and $\iota\colon \{+ : -\} \Rightarrow p$ a natural transformation.

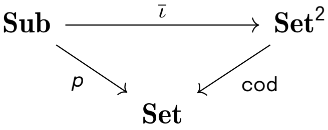
[1] Where $\mathbf{Sub}$ has objects (A, R) with A in $\mathbf{Set}$ and $R\colon A \to 2$ and maps those making the obvious triangle commute. This is the Grothendieck construction associated to P.

Comprehension structures in the literature

The following are all comprehension structures (in order of increasing complexity).



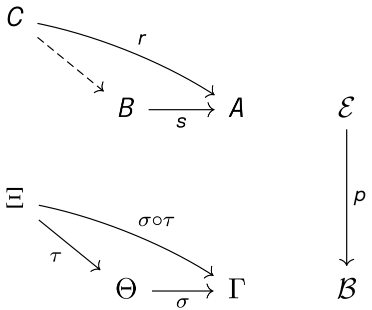
- *comprehension categories* [Jacobs, 1993]: p is a fibration, $\bar{\iota}$ preserves cartesian maps
- *D-categories* [Ehrhard, 1988]: as above, plus a terminal object functor 1 s.t. $1 \dashv \bar{\iota}\mathrm{dom}$
- *doctrine comprehensions* [Lawvere, 1970]: as above, plus p is bifibration



We want to do logic, so we focus on fibrations, but many results apply to generic comprehension structures.

Why fibrations?

Given a functor $p \colon \mathcal{E} \to \mathcal{B}$, s is said to be *p-cartesian* (or *cartesian*) over σ iff $p(s) = \sigma$ and for all r and τ such that $p(r) = \sigma \circ \tau$ there is a unique t such that $r = s \circ t$ and $p(t) = \tau$.
If s is cartesian and over σ, it is said to be a *cartesian lifting* of σ.



Definition ([Grothendieck, 1961])

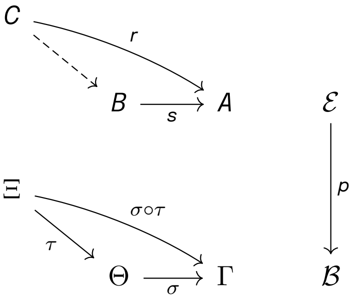
A functor p is a *fibration* iff for all $\sigma \colon \Theta \to pA$ there exists a $s \colon B \to A$ cartesian over σ.

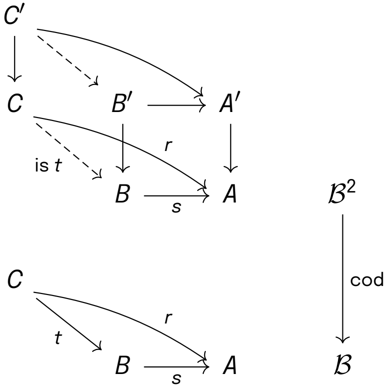
* for the moment "fibration" = "Grothendieck fibration"

What fibrations?

Example (Codomain functor)

Consider the functor $\mathrm{cod}\colon \mathcal{B}^2 \to \mathcal{B}$. A map is cod-cartesian iff it is a pullback in $\mathcal{B}$.





Fibres and reindexing

Definition ([Grothendieck, 1961])

A functor p is a *fibration* iff for all $\sigma : \Theta \to pA$ there exists a $s : B \to A$ cartesian over σ.

It is easy to see that, for a given σ, its lifting is unique up to (vertical) isomorphism.

- For each Γ, we can define a category $\mathcal{E}_\Gamma$ of objects over Γ and maps over id_Γ (called *vertical*), called the *fibre* over Γ.
- For each $\sigma : \Theta \to \Gamma$, existence of the cartesian liftings allows us to move from $\mathcal{E}_\Gamma$ to $\mathcal{E}_\Theta$ (this is not precisely functorial, because of uniqueness up to iso of the lifting!).

$$
\begin{array}{ccccc}
\mathcal{E}_\Theta & \xleftarrow[\sigma^*]{\;-\,-\,-\;} & \mathcal{E}_\Gamma & & \\
\rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} & & \\
A\sigma & \longrightarrow & A & & \mathcal{E} \\
 & & & & \Big\downarrow p \\
\Theta & \xrightarrow[\sigma]{} & \Gamma & & \mathcal{B}
\end{array}
$$

Fibrations and pseudofunctors

Suppose we always have a way to *decide* on a given lifting for each pair (A, σ), that is each fibration comes equipped with a *cleavage*. Then we have the following.

Theorem ([Grothendieck, 1961])

There is a 2-equivalence $\mathbf{Fib}(\mathcal{B}) \cong \mathbf{PsdFun}[\mathcal{B}^{\mathsf{op}}, \mathbf{Cat}]$.

$$\mathbf{Fib}^{split}(\mathcal{B}) \xrightarrow{\qquad\sim\qquad} \mathbf{Fun}[\mathcal{B}^{\mathsf{op}}, \mathbf{Cat}]$$

$$\mathbf{Fib}^{disc}(\mathcal{B}) \xrightarrow{\qquad\sim\qquad} \mathbf{Fun}[\mathcal{B}^{\mathsf{op}}, \mathbf{Set}]$$

$$\mathbf{Fib}^{faith}(\mathcal{B}) \xrightarrow{\sim} \mathbf{Doc}(\mathcal{B}) = \mathbf{Fun}^{\times\text{-}pr}[\mathcal{B}^{\mathsf{op}}, \mathbf{InfSl}]$$

The thing with (non) uniqueness

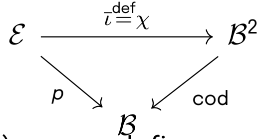
Bien entendu, il y a intérêt le plus souvent à raisonner directement sur des catégories fibrées sans utiliser des clivages explicites, ce qui dispense en particulier de faire appel, pour la notion simple de [...] foncteur cartesién, à une interprétation pesante comme ci-dessus. C'est pour éviter des lourdeurs insupportables, et pour obtenir des énoncés plus intrinsèques, que nous avons dû renoncer à partir de la notion de catégorie clivée [...], qui passe au second rang au profit de celle de catégories fibrée.
Il est d'ailleurs probable que, contrairement à l'usage encore prépondérant maintenant, lié à d'anciennes habitudes de pensée, il finira par s'avérer plus commode dans les problèms universels, de ne pas mettre l'accent sur *une* solution supposée choisie une fois pour toutes mais de mettre toutes les solutions sur un pied d'egalité.

Of course, it is most often useful to reason directly about fibred categories without using explicit cleavages, without the need in particular to appeal, for the simple notion of [...] cartesian functor, to a heavy interpretation as above. It is to avoid unbearable heaviness, and to obtain more intrinsic enunciations, that we had to renounce (or depart) from the notion of split categories [...], which takes second place with respect to that of fibred categories.
It is moreover probable that, contrary to the use still prevalent now, linked to old ways of thinking, it will end up being more convenient for universal problems, not to put the emphasis on a supposed solution chosen once and for all, but to put all solutions on an equal footing.

[Grothendieck, 1961]

The syntactic comprehension category

Recall that a *comprehension category* is a comprehension structure $\{+ : -\}, \iota$ on a functor $p \colon \mathcal{E} \to \mathcal{B}$ such that p is a fibration and $\bar{\iota} \colon \mathcal{E} \to \mathcal{B}^2$ preserves cartesian maps.



Given a notion of type theory (in the sense of [Martin-Löf, 1984]) we can define a comprehension category having:

- $\mathcal{B}_{syn}$ of $\equiv$-equivalence classes of contexts $[\Gamma] = [x_1 : A_1], \dots, [x_n : A_n]$ and maps

$$t \colon [\Theta] \to [\Gamma] \quad \text{iff} \quad \text{for all } i,\ \Theta \vdash t_i : A_i[t_1/x_1, \dots, t_{i-1}/x_{i-1}]$$

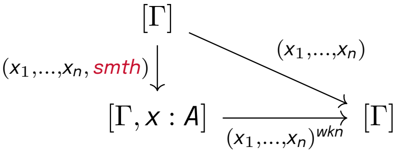
- $\mathcal{E}_{syn}$ of $\equiv$-equivalence classes of typing judgements $[\Gamma \vdash A\ \mathtt{Type}]$ and substitutions

$$(t, s) \colon [\Theta \vdash B\ \mathtt{Type}] \to [\Gamma \vdash A\ \mathtt{Type}] \quad \text{iff} \quad \Theta, y : B \vdash s : A[t/x]$$

- $\chi_{syn} \colon [\Gamma \vdash A\ \mathtt{Type}] \mapsto (\, (x_1, \dots, x_n) \colon [\Gamma, x : A] \to [\Gamma]\,)$ ... and terms?

Terms

We need to find a categorical object corresponding to a term.



$\Gamma \vdash x_1 : A_1$

...

$\Gamma \vdash x_n : A_n[x_1/x_1, \ldots, x_{n-1}/x_{n-1}]$

$\Gamma \vdash smth : A[x_1/x_1, \ldots, x_n/x_n]$

Hence we consider sections of comprehensions.

Interpretation

Recall that a comprehension category is a comprehension structure $\{+ : -\}, \iota$ on a functor $p \colon \mathcal{E} \to \mathcal{B}$ such that p is a fibration and $\bar{\iota} \colon \mathcal{E} \to \mathcal{B}^2$ preserves cartesian maps.

$$\begin{array}{ccc} \mathcal{E} & \xrightarrow{\bar{\iota} \overset{\text{def}}{=} \chi} & \mathcal{B}^2 \\ & {\scriptstyle p} \searrow \quad \swarrow {\scriptstyle \text{cod}} & \\ & \mathcal{B} & \end{array}$$

$$\begin{array}{ccc} \Theta.B & \xrightarrow{\bar{\sigma}} & \Gamma.A \\ {\scriptstyle \chi_B} \downarrow & \lrcorner & \downarrow {\scriptstyle \chi_A} \\ \Theta & \xrightarrow{\sigma} & \Gamma \end{array}$$

$$B \xrightarrow{cart} A$$

$$\Theta \xrightarrow{\sigma} \Gamma$$

objects of $\mathcal{B}$	contexts
objects of $\mathcal{E}$	types in context
χ_A	comprehension/context extension
$\text{dom}\chi_A = \Gamma.A$	extended context
sections of χ_A	terms of type A
pullback	substitution

Admissible rules

What rules are admissible here?

$$\frac{\vdash \Gamma, x : A, \Delta \ \mathtt{ctx}}{\Gamma, x : A, \Delta \vdash x : A}(\mathsf{Var}) \qquad \frac{\Gamma \vdash a : A \quad \Gamma, x : A, \Delta \vdash \mathcal{J}}{\Gamma, \Delta[a/x] \vdash \mathcal{J}[a/x]}(\mathsf{Sbst}) \qquad \frac{\Gamma \vdash A \ \mathtt{Type} \quad \Gamma, \Delta \vdash \mathcal{J}}{\Gamma, x : A, \Delta \vdash \mathcal{J}}(\mathsf{Wkn})$$

for $\mathcal{J} ::= \Gamma \vdash A \ \mathtt{Type}, \Gamma \vdash A = A' \ \mathtt{Type}, \Gamma \vdash a : A, \Gamma \vdash a = a' : A$,
plus classical rules for definitional equality, see [Hofmann, 1997].
Let's see how.

Remark. Existence of the (unique up to iso) cartesian lifting of σ at A induces a suitable pullback. We might denote $B = A\sigma$ in this case, but mind that (if we had to have one) this forgets our choice!

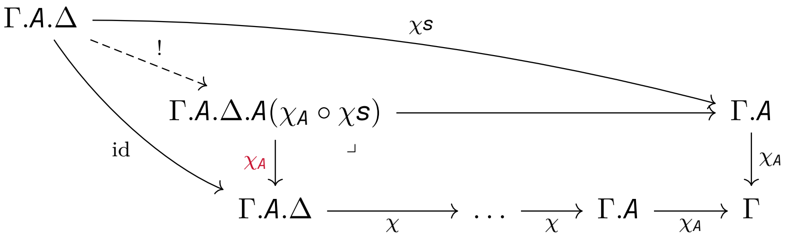
$$\begin{array}{ccc} \Theta.B & \xrightarrow{\bar{\sigma}} & \Gamma.A \\ {\scriptstyle \chi_B}\downarrow & \lrcorner & \downarrow{\scriptstyle \chi_A} \\ \Theta & \xrightarrow{\sigma} & \Gamma \end{array}$$

$$B \xrightarrow{cart} A$$

$$\Theta \xrightarrow{\sigma} \Gamma$$

The rule (Var)

$$\frac{\vdash \Gamma, x : A, \Delta \ \mathtt{ctx}}{\Gamma, x : A, \Delta \vdash x : A} \text{(Var)}$$

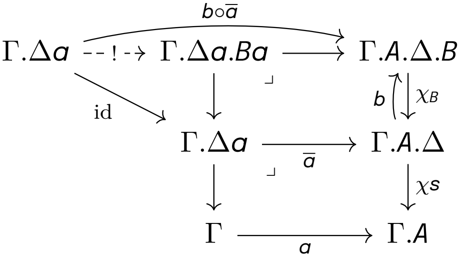


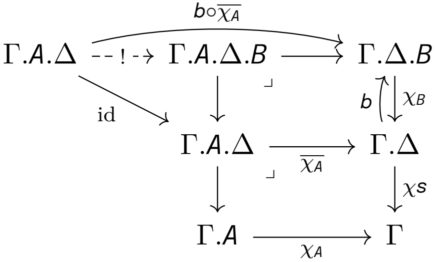
The rules (Sbst) and (Wkn)

for $\mathcal{J} = b : B$

$$\frac{\Gamma \vdash a : A \quad \Gamma, x : A, \Delta \vdash b : B}{\Gamma, \Delta[a/x] \vdash b[a/x] : B[a/x]}\text{(Sbst)}$$

$$\frac{\Gamma \vdash A\,\mathrm{Type} \quad \Gamma, \Delta \vdash b : B}{\Gamma, x : A, \Delta \vdash b : B}\text{(Wkn)}$$





Another kind of model

A simple definition hides a lot of structure. Another perspective is that of categories with families.

Definition (Cwf, [Dybjer, 1996])

A *category with families* is the data of

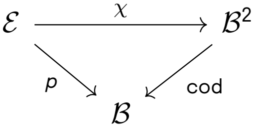
- a category $\mathcal{B}$ with terminal object $\top$;
- a functor $F = (\mathsf{Ty}, \mathsf{Tm}) \colon \mathcal{B}^{\mathsf{op}} \to \mathbf{Fam}$, with $\mathbf{Fam}$ of set-indexed sets;
- for each Γ in $\mathcal{B}$ and A in $\mathsf{Ty}(\Gamma)$ an object $\Gamma.A$ in $\mathcal{B}$, together with two projections $\mathsf{p}_A \colon \Gamma.A \to \Gamma$ and $\mathsf{v}_A \in \mathsf{Tm}(\Gamma.A, \mathsf{Ty}\,\mathsf{p}_A(A))$ such that for each $\sigma \colon \Theta \to \Gamma$ and $a \in \mathsf{Tm}(\mathsf{Ty}\sigma(A))$ there exists a unique morphism $\Theta \to \Gamma.A$ making the obvious triangles commute.

$$F(\Gamma) = (\ \mathsf{Ty}(\Gamma), (\mathsf{Tm}(\Gamma, A))_{A \in \mathsf{Ty}(\Gamma)}\ )$$

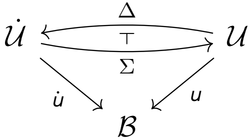
A discrete equivalence

Theorem (Cartmell, Moggi, Hofmann, Dybjer, Awodey)

Cwfs are equivalent to comprehension categories with p discrete.



p discrete
[Jacobs, 1993]



$u, \dot{u}$ discrete
[Awodey, 2018]

$$\mathrm{Ty} : \mathcal{B}^{\mathrm{op}} \to \mathbf{Set}$$

$$\mathrm{Tm} : \left(\int \mathrm{Ty} \right)^{\mathrm{op}} \to \mathbf{Set}$$

[Dybjer, 1996]

A general biequivalence

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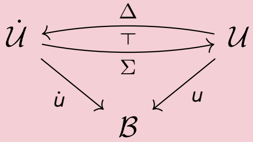
We extend the result to a biequivalence involving more than just discrete fibrations.

- Non-discrete: so that we can talk "syntactically" about theories where $\mathcal{E}_\Gamma$ is more than a set.
  $\rightsquigarrow$ *e.g.* subtyping
- Biequivalence: so that we can learn a lesson from doctrines and manipulate the notion of model, describe model morphisms and so on.
  $\rightsquigarrow$ internalizing allows us to do more stuff

Generalized cwfs

Definition [C.-Di Liberti, 2022]

A *generalized category with families* (or *judgemental dtt*) is the data of two fibrations $u, \dot{u}$, a functor Σ making the triangle commute and preserving cartesian maps (*i.e.* a 1-cell in $\mathbf{Fib}$), Δ right adjoint to Σ with cartesian unit and counit.

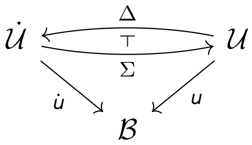


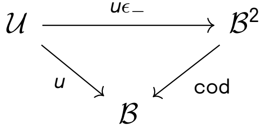
As in the discrete case, $\mathcal{U}$ collects types (in contexts), $\dot{\mathcal{U}}$ terms (fibred over types and contexts), Σ performs typing, $\Delta\colon (\Gamma \vdash A\ \mathtt{Type}) \mapsto (\Gamma.A \vdash x : A)$.

Comparing compcats and gcwfs

Proposition [C.-Emmenegger, 2023]

A compcat induces a gcwf, and viceversa.





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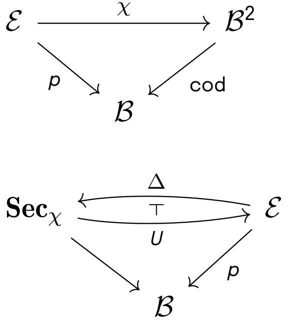
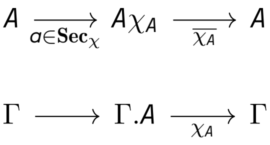
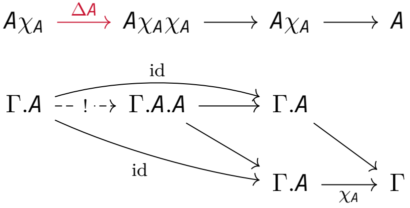






Does this ring any bells?

It's all comonads!

When trying to compare the two, one quickly notices the ubiquity of comonads:

- a gcwf is defined as an adjunction, hence we always have a comonad $\Sigma\Delta$,
- given a compcat, we can use comprehensions to define a kernel-pair-like comonad.

Definition [Jacobs, 1999]

Let $p \colon \mathcal{E} \to \mathcal{B}$ a fibration. A *weakening and contraction comonad* on p is a comonad (K, ϵ, ν) on $\mathcal{E}$ with ϵ cartesian and for each cartesian map in $\mathcal{E}$ its naturality square is a pullback.

Remark. They are equivalent to comprehension categories.

Weakening and contraction comonads

Definition [Jacobs, 1999]

Let $p \colon \mathcal{E} \to \mathcal{B}$ a fibration. A *weakening and contraction comonad* on p is a comonad (K, ϵ, ν) on $\mathcal{E}$ with ϵ cartesian and for each cartesian map in $\mathcal{E}$ its naturality square is a pullback.

$$K \circlearrowright \mathcal{E} \xrightarrow{\;p\;} \mathcal{B}$$

$KA = A\chi_A$ models extension
$\epsilon \colon K \Rightarrow \mathrm{Id}$ models weakening
$\nu \colon K \Rightarrow KK$ models contraction

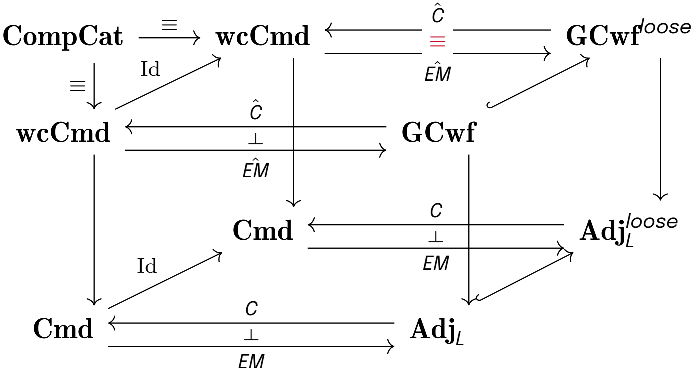
$\Gamma.A \vdash A\ \mathtt{Type}$
from $\Gamma \vdash B\ \mathtt{Type}$ to $\Gamma.A \vdash B\ \mathtt{Type}$
from $\Gamma.A.A \vdash B\ \mathtt{Type}$ to $\Gamma.A \vdash B\ \mathtt{Type}$

Now to 2-categories

We use the 2-categorical structure of both **Fib** and **Cmd**!

Theorem [C.-Emmenegger, 2023]

The classical comonad-adjunction adjunction lifts as follows.



$\mathbf{Cmd}$ has 0-cells $(\mathcal{C}, K, \epsilon, \nu)$

1-cells $(H, \theta)\colon (\mathcal{C}, K, \epsilon, \nu) \to (\mathcal{C}', K', \epsilon', \nu')$ with $H\colon \mathcal{C} \to \mathcal{C}'$ and $\theta\colon HK \Rightarrow K'H$

s.t. $\epsilon' H * \theta = H\epsilon$, $\nu' H * \theta = K'\theta * \theta K * H\nu$

2-cells $\phi\colon (H_1, \theta_1) \Rightarrow (H_2, \theta_2)$ is $\phi\colon H_1 \Rightarrow H_2$

s.t. $(K'\phi)\theta_1 = \theta_2(\phi K)$

$\mathbf{wcCmd}$ has 0-cells $(p, \mathcal{C}, K, \epsilon, \nu)$

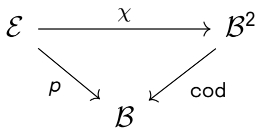
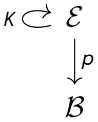
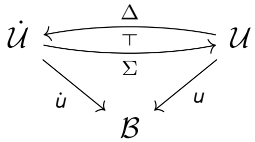
1-cells $(H, \theta, C)\colon (p, \mathcal{C}, K, \epsilon, \nu) \to (p', \mathcal{C}', K', \epsilon', \nu')$

with (H, θ) a 1-cell in $\mathbf{Cmd}$ and (H, C) a 1-cell in $\mathbf{Fib}$

2-cells $(\phi, \psi)\colon (H_1, \theta_1, C_1) \Rightarrow (H_2, \theta_2, C_2)$

with ϕ a 2-cell in $\mathbf{Cmd}$ and (ϕ, ψ) a 2-cell in $\mathbf{Fib}$

Recap

Type theory			
contexts	$\mathrm{Ob}(\mathcal{B})$	$\mathrm{Ob}(\mathcal{B})$	$\mathrm{Ob}(\mathcal{B})$
types	$\mathrm{Ob}(\mathcal{E})$	$\mathrm{Ob}(\mathcal{E})$	$\mathrm{Ob}(\mathcal{U})$
$\Gamma \vdash A\ \mathrm{Type}$	$pA = \Gamma$	$pA = \Gamma$	$uA = \Gamma$
$\Gamma.A \to \Gamma$	χ_A	$p\epsilon_A$	$u\epsilon_A$
$\Gamma.A$	$\mathrm{dom}(\chi_A)$	pKA	$u\Sigma\Delta A$
A^+ (A in $\Gamma.A$)	$A\chi_A$	KA	$\Sigma\Delta A$
terms	sections	sections	$\mathrm{Ob}(\dot{\mathcal{U}})$
$\Gamma \vdash a : A$	section of χ_A	section of ϵ_A	$\Sigma a = A$

A lesson from doctrines

In $\mathbf{Doc}$ the 2-category of doctrines we have $LT_{\mathcal{T}}$ the "syntactic" doctrine of a given theory and S, the subset doctrine.

Lemma

$$1\text{-cells in } \mathbf{Doc}\ LT_{\mathcal{T}} \to S \quad \leftrightarrow \quad \text{set-based models of } \mathcal{T}$$

$$2\text{-cells in } \mathbf{Doc}\ LT_{\mathcal{T}} \underset{\longrightarrow}{\overset{\longrightarrow}{\Downarrow}} S \quad \leftrightarrow \quad \text{morphisms of set-based models of } \mathcal{T}$$

Intuitively, we map a variable the the set of its extension, and a formula to the subset where it is true. Both doctrines encode the structure needed, the maps from one to the other describes the process of modeling something with something else.

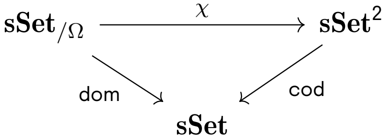
Learning the lesson

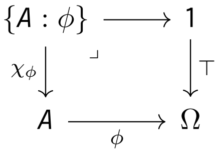
Ambient 2-category	syntactic object	semantic object
Doc	$LT_{\mathcal{T}}$	subset doctrine
CompCat	χ_{syn}	???

What candidates are we interested in for the role of the semantic object?
Historically, simplicial sets, so let us start there...

The topos comprehension on $\mathbf{sSet}$

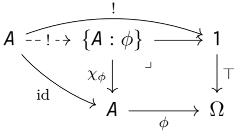
Recall that $\mathbf{sSet} = \mathbf{PSh}(\mathbb{\Delta})$, with $\mathbb{\Delta}$ the simplex category, is a (Grothendieck) topos. Each topos induces a comprehension category via its subobject classifier, in this case Ω is $\Omega_n = \{\text{sieves on } [n]\}$ and $\top : 1 \to \Omega$ picks out the maximal sieve.





This is not quite right, why?

It is proof irrelevant!



Voevodsky's model I

[Voevodsky, 2015, Kapulkin and Lumsdaine, 2021] describe how to get from $\mathbf{sSet}$, for a given inaccessible cardinal α, a contextual category $\mathcal{C}_\alpha$ (such that for $\beta < \alpha$ inaccessible it contains a universe U_β satisfying UA).

1. $\mathbf{W}_\alpha : \mathbf{sSet}^{\mathrm{op}} \to \mathbf{Set}$, $\mathbf{W}_\alpha(X) = \{$isos classes of α-small well ordered morphisms into $X\}$
2. $\mathbf{W}_\alpha$ is representable and represented by a W_α in $\mathbf{sSet}$
3. call $q_\alpha : \tilde{W}_\alpha \to W_\alpha$ the $\mathbf{sSet}$ morphism associated to id_{W_α}
4. consider $\mathbf{U}_\alpha \hookrightarrow \mathbf{W}_\alpha$ of morphisms that are Kan fibrations[2], represented by an U_α
5. $$\begin{array}{ccc} \tilde{U}_\alpha & \longrightarrow & \tilde{W}_\alpha \\ {\scriptstyle p_\alpha \overset{\text{def}}{=}}\big\downarrow & \lrcorner & \big\downarrow {\scriptstyle q_\alpha} \\ U_\alpha & \hookrightarrow & W_\alpha \end{array} \qquad \begin{array}{c} \mathbf{U}_\alpha(X) \\ \wr\big\downarrow \\ \mathbf{sSet}(X, U_\alpha) \end{array} \qquad \begin{array}{c} F \overset{\langle f \rangle}{\twoheadrightarrow} X \\ \\ X \overset{f}{\to} U_\alpha \end{array}$$

[2] fibrations wrt the standard model category on $\mathbf{sSet}$

Voevodsky's model II

6. p_α is a universe in $\mathbf{sSet}$ *i.e.* a choice of pb exists

$$\begin{array}{ccc} (X;f) & \xrightarrow{Q(f)} & \tilde{U}_\alpha \\ {\scriptstyle P(X,f)}\downarrow & \lrcorner & \downarrow{\scriptstyle p_\alpha} \\ X & \xrightarrow{f} & U_\alpha \end{array}$$

We construct (the split comprehension category associated to) $\mathcal{C}_\alpha$.

$$\begin{array}{ccc} \mathcal{T}_\alpha & \xrightarrow{\chi_\alpha} & \mathcal{C}_\alpha^2 \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle \mathrm{cod}} & \\ & \mathcal{C}_\alpha & \end{array}$$

$$\begin{aligned}
(\mathcal{C}_\alpha)_n &= \{\underline{f}_n = (f_1, \dots, f_n) \in (\mathit{Mors}\mathbf{Set})^n \mid f_i \colon (1; f_1, \dots, f_{i-1}) \to U_\alpha\} \\
\mathcal{C}_\alpha(\underline{f}_n, \underline{g}_m) &= \mathbf{sSet}(\,(1; f_1, \dots, f_n), (1; g_1, \dots, g_m)) \\
(\mathcal{T}_\alpha)_n &= \{f \text{ map in } \mathcal{C}_\alpha \text{ s.t. } f \colon \underline{f}_{m+n} \to \underline{f}_m \text{ and } f = P \circ \cdots \circ P\} \\
\mathcal{T}_\alpha(f, g) &= \{\text{squares in } \mathcal{C}_\alpha\} \\
\chi(f) &= f
\end{aligned}$$

Lemma

For each α inaccessible there is a 1-cell $\chi_{syn} \to \chi_\alpha$ in $\mathbf{CompCat}^{split}$.

Proof. By structural induction, given that $\mathcal{C}_\alpha$ is a contextual category.

Kan fibrations

What if we directly use Kan fibrations?
In fact, their inclusion into $\mathbf{sSet}^2$ induces a (non split) compcat,

$$\begin{array}{ccc} \mathbf{Kan} & \xrightarrow{\ \chi_K\ } & \mathbf{sSet}^2 \\ & {\scriptstyle \mathsf{cod}} \searrow \quad \swarrow {\scriptstyle \mathsf{cod}} & \\ & \mathbf{sSet} & \end{array}$$

moreover

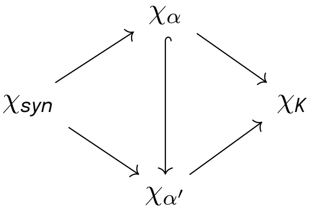
Lemma

For each α inaccessible there is a 1-cell $\chi_\alpha \to \chi_K$ in $\mathbf{CompCat}^{full}$.

Proof. Map $(f_1, \ldots, f_n)$ to $(\,(1; f_1, \ldots, f_{n-1}); f_n\,)$.

Is this it?

For $\alpha \leq \alpha'$ inaccessible, we have the following in $\mathbf{CompCat}$.



Question 1. How much is χ_K "the" semantic object for MLTT? For example, can we build out of a generic $\chi_{syn} \to \chi_K$ a contextual category?
Question 2. What other object might we be interested in considering?

Thank you for listening!

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