Rigidification of cartesian closed ∞-functors

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Outline

The internal language conjectures

Conjecture

The functor $\mathbf{Ho}_\infty$*, turning a relative category into a quasicategory, restricts to DK-equivalences*

$$\iota : \mathbf{CompCat}_{\Sigma, Id} \to \mathbf{Qcat}_{lex}$$

$$\iota_\pi : \mathbf{CompCat}_{\Sigma, \Pi_{ext}, Id} \to \mathbf{Qcat}_{lcc}$$

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where

- The domain categories have morphisms that preserves the structure involved up to isomorphism.
- The codomain categories have morphisms that preserves the structure involved up to equivalence.

With Π-types

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In [1], we described an object-wise construction to get, from a lcc quasicategory $\mathcal{C}$, a π-tribe $\mathcal{T}$ (with its canonical comprehension structure) so that $\mathbf{Ho}_\infty(\mathcal{T}) \simeq \mathcal{C}$.

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This shows that the functor induced by ι_π on homotopy categories is essentially surjective on objects. We then claimed mistakenly that ι being fully faithful (as an ∞-functor) implied directly the same for ι_π, and hence the second part of the conjecture.

The mistake

However, the argument overlooked the mismatch between the structure preservation expected from the morphisms

- up-to-isomorphism in $\mathbf{CompCat}_{\Sigma,\Pi_{\mathrm{ext}},\mathrm{Id}}$
- up-to-equivalence in $\mathbf{Qcat}_{lcc}$

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One should also find a way to rigidify the lcc ∞-functors in $\mathbf{Qcat}_{lcc}$.

Outline

Clans

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Definition

A clan structure on a category $\mathcal{C}$ with a terminal object $\mathbf{1}$ is given by a class of maps $\mathcal{F}$ called fibrations such that:

- Isomorphims are fibrations, $X \to \mathbf{1}$ is a fibration for every X.
- Fibration are closed under composition. Pullbacks of fibrations exists and yield fibrations.

Tribes

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We will also consider π-tribes, which are essentially tribes such that every fibration admits an *internal product* along any fibration.
Essentially, this means that the underlying type theory also has Π-types.

Canonical comprehension

Given a π-tribe $\mathcal{T}$, the canonical comprehension structure given by the Grothendieck fibration

$$\textbf{cod} : \mathcal{T}^{\rightarrow}_{\mathsf{fib}} \to \mathcal{T}$$

supports Σ and Π types that are stable under pullback up to isomorphism.

Substitution is well-defined and functorial up to isomorphism.

Strictification

A **strictification** procedure aims at replacing this comprehension category by an equivalent split one (the splitting is meant to include the Π-types choices).

Pictorially:

$$\text{Isomorphisms} \quad \Longrightarrow \quad \text{Equalities}$$

Cohenrece in an ∞-category

Given a locally cartesian closed $(\infty,1)$-category $\mathcal{C}$ (e.g. a quasicategory) with a terminal object, pullbacks give a substitution operation which is well-defined and functorial up to (homotopy) equivalence. Internal products are also defined (and pullback-stable) up to equivalence.

Rigidification

A **rigidification** procedure aims at replacing $\mathcal{C}$ by a π-tribe presenting the same (∞,1)-category (up to equivalence).

Pictorially:

$$\text{Equivalences} \qquad \Longrightarrow \qquad \text{Isomorphisms}$$

Outline

The approach for ι

In [5], the authors' approach for proving the first part of the conjecture could be summed up as:

- Factor ι as

$$\mathbf{CompCat}_{\Sigma,\mathsf{Id}} \to \mathbf{Trb} \to \mathbf{Qcat}_{lex}$$

- Observe that the canonical functor

$$\mathbf{Trb} \to \mathbf{CompCat}_{\Sigma,\mathsf{Id}}$$

  is a homotopy inverse to

$$\mathbf{CompCat}_{\Sigma,\mathsf{Id}} \to \mathbf{Trb}$$

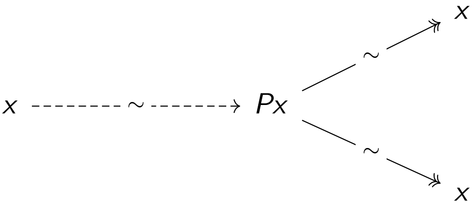
The approach for ι

- Find a subcategory **sTrb** of **Trb**, equivalent to **Trb** as a relative category, that can that be equipped with a fibration category structure.

- Construct a functorial rigidification functor $\mathbf{Qcat}_{lex} \to \mathbf{sTrb}$ whose derived functor is inverse equivalence to (the derived functor of) $\mathbf{sTrb} \to \mathbf{Qcat}_{lex}$.

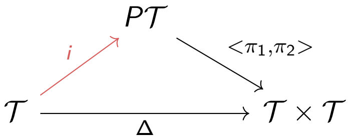
Fibration category of tribes

Trb can be equipped with a notion of fibration making it "almost" a fibration category in that, given a tribe $\mathcal{T}$, there is a canonical tribe $P\mathcal{T}$ whose objects are essentially the spans of trivial fibration in $\mathcal{T}$.
For every object x, we may chose a path object Px (in $\mathcal{T}$) for x:



Fibration category of tribes

We then have a mapping $i : \mathcal{T} \to P\mathcal{T}$ fitting in a commutative triangle



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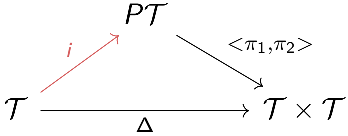

However, the choice made need not imply that the mapping i is functorial. This is how we fall short of constructing a path object for $\mathcal{T}$, the only thing left needed for **Trb** to be a fibration category.

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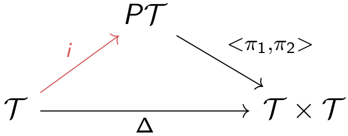


However, the choice made need not imply that the mapping i is functorial. This is how we fall short of constructing a path object for $\mathcal{T}$, the only thing left needed for **Trb** to be a fibration category.
If $\mathcal{T}$ is a semi-simplicial tribe, taking $Px := x^{\Delta^1}$ makes i functorial.

The approach for ι_π

To prove the second part of the conjecture, our approach is the following:

- Factor ι_π as

$$\mathbf{CompCat}_{\Sigma,\Pi_{\mathrm{ext}},\mathrm{Id}} \to \mathbf{hTrb}_\pi \to \mathbf{Qcat}_{lcc}$$

  where $\mathbf{hTrb}_\pi$ is a full subcategory of $\mathbf{Trb}_\pi$ admitting a fibration category structure.
- Observe that the canonical functor

$$\mathbf{hTrb}_\pi \to \mathbf{CompCat}_{\Sigma,\Pi_{\mathrm{ext}},\mathrm{Id}}$$

  is a homotopy inverse to the first inclusion.

The approach for ι_π

- Check that (the second inclusion in the factorization of) ι restricts to a DK-equivalence

$$\mathbf{hTrb}_\pi^\sim \to \mathbf{Qcat}_{lcc}$$

  where $\mathbf{hTrb}_\pi^\sim$ is a version of $\mathbf{hTrb}_\pi$ with morphisms preserving internal product up-to-equivalence.

- Provide a rigidification procedure on morphims to prove that the inclusion $\mathbf{hTrb}_\pi \to \mathbf{hTrb}_\pi^\sim$ is a DK-equivalence.

The rigidification tool

Lemma

Consider a morphism $f : \mathcal{T} \to \mathcal{S}$ in $\mathbf{hTrb}_{\pi}^{\sim}$ and form the following pullback square:

$$\begin{array}{ccc} \mathcal{T}' & \xrightarrow{\quad u \quad} & P\mathcal{S} \\ \downarrow & \lrcorner & \downarrow \\ \mathcal{T} \times \mathcal{S} & \xrightarrow[f \times id_{\mathcal{S}}]{} & \mathcal{S} \times \mathcal{S} \end{array}$$

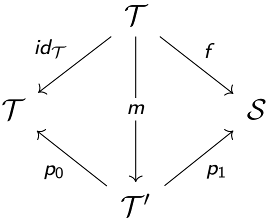
Then $\mathcal{T}'$ is a π-tribe equivalent to $\mathcal{T}$ and the morphisms $\mathcal{T}' \to \mathcal{T}$ and $\mathcal{T}' \to \mathcal{S}$ are π-closed.

The rigidification tool at work

Corollary

$\mathbf{Ho}(\mathbf{hTrb}_\pi) \to \mathbf{Ho}(\mathbf{hTrb}_\pi^\sim)$ *is full.*

Proof.



implies $[f] = [p_1] \circ [p_0]^{-1}$ in $\mathbf{Ho}(\mathbf{hTrb}_\pi^\sim)$ □

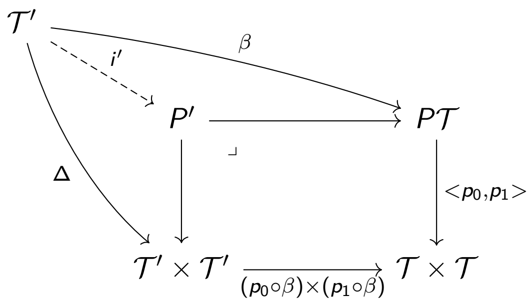
The rigidification tool at work

Corollary

$\mathbf{Ho}(\mathbf{hTrb}_\pi) \to \mathbf{Ho}(\mathbf{hTrb}^{\sim}_\pi)$ *is essentially surjective on objects.*

Proof.

Taking $\mathcal{S} = P\mathcal{T}$ and $f := i : \mathcal{T} \to P\mathcal{T}$ the morphism into the path object:



□

Wrapping up

$$\mathbf{Ho}(\mathbf{hTrb}_{\pi}) \to \mathbf{Ho}(\mathbf{hTrb}_{\pi}^{\sim})$$

is clearly faithful so that

$$\iota_{\pi} : \mathbf{CompCat}_{\Sigma,\Pi_{\mathrm{ext}},\mathrm{Id}} \to \mathbf{hTrb}_{\pi} \to \mathbf{hTrb}_{\pi}^{\sim} \to \mathbf{Qcat}_{lcc}$$

is a composite of three DK-equivalences.

Thank you for you attention!

Some fibration categories of tribes

- **sTrb** the category of semi-simplicial tribes (as defined in [5])
- **hTrb** the full subcategory (of **Trb**) formed by those tribes $\mathcal{T}$ such there exists a morphism of tribe $\iota : \mathcal{T} \to P\mathcal{T}$ which composes with the projection $P\mathcal{T} \to \mathcal{T} \times \mathcal{T}$ to yield the diagonal $\mathcal{T} \to \mathcal{T} \times \mathcal{T}$

Some fibration categories of tribes

- $\mathbf{hTrb}_\pi$ the full subcategory (of $\mathbf{Trb}_\pi$) formed by those π-tribes $\mathcal{T}$ admitting a path object.
- $\mathbf{hTrb}_\pi^\sim$ the (non-full) subcategory (of $\mathbf{hTrb}$) consisting in the tribes which are equivalent to a π-tribe, and with morphisms between them the morphisms of tribes $m : \mathcal{T} \to \mathcal{T}'$ such that $\mathbf{Ho}_\infty(m)$ preserves the structure of locally cartesian closed quasicategories (i.e. $\mathbf{Ho}_\infty(m)$ is a cartesian closed ∞-functor).

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