On epimorphisms and acyclic types in univalent mathematics

Ulrik Buchholtz[1]
jww/ Tom de Jong[1] and Egbert Rijke[2]

School of Computer Science, University of Nottingham

Department of Mathematics, University of Ljubljana

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Epimorphisms and acyclic maps

Definition (Epimorphism)

A map $f : A \to B$ is an *epimorphism* if for every type X, the precomposition map

$$(B \to X) \xrightarrow{f^*} (A \to X)$$

is an embedding.

So extensions along epimorphisms are unique if they exist.

Definition (Acyclicity)

A type is *acyclic* if its suspension is contractible; a map is *acyclic* if all its fibers acyclic.

Lemma

A map $f : A \to B$ is epic if and only if the square

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ f\downarrow & & \downarrow\mathrm{id} \\ B & \xrightarrow[\mathrm{id}]{} & B \end{array}$$

is a pushout.

Lemma

For all $f : A \to B$, $b : B$: $\mathrm{fib}_{\nabla_f}(b) \simeq \Sigma\, \mathrm{fib}_f(b)$.

Corollary

A map is epic if and only if it is acyclic.

Hatcher's 2-dimensional example

Proposition

Acyclic types are connected with perfect fundamental groups.

Proposition

The unit type is an acyclic type, and equivalences are acyclic maps.

A non-trivial example of an acyclic type can by found in Hatcher's book [Hat02, Ex. 2.38]: We import this as the higher inductive type X with constructors:

$$\mathrm{pt} : X,\ a, b : \Omega X,\ r : a^5 = b^3,\ s : b^3 = (ab)^2$$

Proposition

The type X is acyclic.

Proof.

ΣX is the HIT with constructors $\mathrm{N}, \mathrm{S} : \Sigma X$, $\mathrm{m}_{\mathrm{pt}} : \mathrm{N} = \mathrm{S}$, $a, b : \mathrm{m}_{\mathrm{pt}} = \mathrm{m}_{\mathrm{pt}}$, $r : a^5 = b^3$, $s : b^3 = (ab)^2$.
Contract away $(\mathrm{S}, \mathrm{m}_{\mathrm{pt}})$, then use Eckmann–Hilton to re-express s as $s : b = a^2$. Then contract away (b, s), so $r : a^5 = a^6$, equivalently, $r : a = \mathrm{refl}$. Finally, contract away (a, r), leaving the unit type. □

Interpreting a as a 5-cycle and b as a 3-cycle, we get a 0-connected map $X \to \mathrm{BA}_5$.

Closure properties

Proposition

If $f\colon A \to B$ is acyclic, then $g\colon B \to C$ is acyclic if and only if $g \circ f$ is.

Proposition

Acyclic maps are closed under pullbacks and pushouts along arbitrary maps.

Proposition

The acyclic maps are stable under composition, retracts, finite products, and all coproducts and pushouts in the arrow category.

Acyclic types are not closed under coproducts ($\mathbf{1} + \mathbf{1}$ is not acyclic), nor identity types, nor truncations.

Classically, the acyclic maps form left class of an orthogonal factorization system, via Quillen's plus construction, $X \to X^+$, an acyclic map killing the perfect core of $\pi_1 X$.

$\rightsquigarrow$ Algebraic K-Theory

The Volodin space $X(R)$ of a ring R is the fiber of $\mathrm{BGL}(R) \to \mathrm{BGL}(R)^+$, and hence acyclic. We have $\pi_1 X(R) \cong \mathrm{St}(R)$, the *Steinberg group*. This is not acyclic, since $\mathrm{H}_3(\mathrm{St}(R)) \cong \mathrm{K}_3(R)$, [Wei13, Ex. 1.9].

(Quillen) $\mathrm{K}_3(\mathbb{F}_q) \cong \mathbb{Z}/(q^2 - 1)$, so the 1-truncation of an acyclic type need not be acyclic.

The plus principle

We don't know whether the factorization system can be constructed in HoTT. Many properties seem to need further axioms, like:

Plus Principle (PP)

Every acyclic and simply connected type is contractible.

Hoyois highlighted this in the context of Grothendieck $(\infty, 1)$-toposes [Hoy19, Rem. 4]. It's open whether it holds in all such; (PP) holds in parametrized spectra and follows from Whitehead's Principle (WP) via:

Lemma

Any acyclic and simply connected type is infinitely connected.

Lemma (PP)

Any acyclic 1-equivalence is an equivalence.

Proposition (PP)

Let $f : A \to B$ be acyclic, and let $f' : A \to X$ be any map. Then f' extends along f if (and only if) $\ker(\pi_1(f)) \subseteq \ker(\pi_1(f'))$.

Proof.

Form the pushout:

$$\begin{array}{ccc} A & \xrightarrow{f'} & X \\ f\downarrow & \ulcorner & \downarrow g \\ B & \longrightarrow & P \end{array}$$

Then g is an acyclic 1-equivalence. □

The Higman group

An interesting acyclic 1-type is the classifying type of the Higman group [Hig51]:

$$\mathrm{H} = \langle\, a, b, c, d \mid a = [d, a], b = [a, b], \\ c = [b, c], d = [c, d]\,\rangle.$$

Let BH be the HIT for the presentation complex, with nine constructors and no truncation.

Proposition

The type BH *is acyclic.*

To show $\mathrm{BH} \not\simeq \mathbf{1}$, we use the following [Wär23].

Theorem (Wärn)

If $A \leftarrow R \rightarrow B$ is a span of 0-truncated maps of 1-types, then the pushout $A +_R B$ is a 1-type and the inclusion maps are 0-truncated.

Let $\mathrm{B}\langle x_i \rangle$ be the sub-HIT of BH using only constructors involving the x_i. Then:

$$\begin{array}{ccc} \mathrm{B}\langle b\rangle & \longrightarrow & \mathrm{B}\langle b, c\rangle \\ \downarrow & \ulcorner & \downarrow \\ \mathrm{B}\langle a, b\rangle & \longrightarrow & \mathrm{B}\langle a, b, c\rangle \end{array} \qquad \begin{array}{ccc} \mathrm{B}\langle a, c\rangle & \longrightarrow & \mathrm{B}\langle a, b, c\rangle \\ \downarrow & \ulcorner & \downarrow \\ \mathrm{B}\langle c, d, a\rangle & \longrightarrow & \mathrm{BH} \end{array}$$

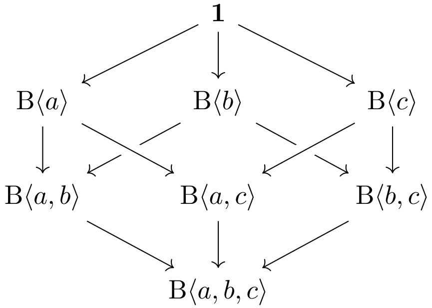
And $\mathrm{B}\langle a, b\rangle$ is an HNN-extension (i.e., coequalizer of groupoids):

$$\mathbb{S}^1 \underset{b^2}{\overset{b}{\rightrightarrows}} \mathrm{B}\langle b\rangle \longrightarrow \mathrm{B}\langle a, b\rangle$$

So $\mathrm{B}\langle a, b\rangle$ (with π_1 a Baumslag–Solitar group) is a 1-type and $\mathrm{B}\langle b\rangle \rightarrow \mathrm{B}\langle a, b\rangle$ is 0-truncated. We have section/retraction $\mathrm{B}\langle a\rangle \rightleftarrows \mathrm{B}\langle a, b\rangle$, so the other inclusion is 0-truncated, too.

The Higman group, continued

It remains to see that the maps of the form $\mathrm{B}\langle a, c\rangle \to \mathrm{B}\langle a, b, c\rangle$ are 0-truncated. The follows by descent from looking at the commuting cube:



The top and bottom faces are pushouts, and the back faces are pullbacks, so the front faces are pullbacks as well. Since the front bottom maps are (individually and jointly) surjective, and the maps on the sides are 0-truncated, the map in front is as well, as desired.

NB This proof completely avoids classical combinatorial group theory!

Outlook & Thanks

- Using (PP), a map is acyclic if and only if it is *balanced* [Rap19].
- This implies that the fiber sequence of an acyclic map of connected types is a cofiber sequence.
- Again with (PP), we have $A \perp X$ for acyclic A and *hypoabelian* X (i.e., $\pi_1 X$ has no perfect subgroups).
- Outright, we have $A \perp X$ for acyclic X and nilpotent X that are limits of their Postnikov towers.
- With (WP), a type is acyclic if and only if its integral homology is trivial.
- We also study k-epimorphisms and k-acyclic types.
- We can construct (a candidate for) BA_5^+ – correct assuming (WP), what about without?
- We believe that plus-constructions can always be performed assuming (WP), Sets Cover (SC), and Countable Choice (CC).
- Also to do: look at acyclicity of $\mathrm{BAut}(\mathbb{N})$ and $\mathrm{B}\Sigma_\infty \to Q_0\mathbb{S}^0 \simeq \mathrm{B}\Sigma_\infty^+$.

Thank you!

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